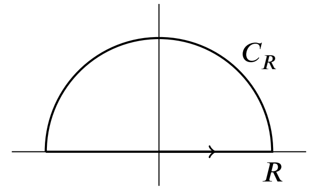
Quiz

1. Evaluate $\displaystyle\int_{C_R} \frac{e^{iz}}{(z^2+1)^2}\,dz$ by using the residue theorem (notice the double pole at i). Deduce that



$$\int_0^\infty \frac{\cos(x)}{(x^2+1)^2}\,dx = \frac{\pi}{2e}.$$

Solution. Here is a computation of the residue of the integrand at i:

$$\frac{d}{dz}\left(\frac{e^{iz}}{(z+i)^2}\right)\Bigg|_{z=i} = \left(\frac{ie^{iz}}{(z+i)^2} - \frac{2e^{iz}}{(z+i)^3}\right)\Bigg|_{z=i} = \frac{e^{-1}}{(2i)^2}\left(i - \frac{2}{2i}\right) = \frac{1}{2ie}.$$

As long as $R > 1$, so that the singular point i is inside the curve, the value of the integral is $2\pi i$ times the residue, or π/e.

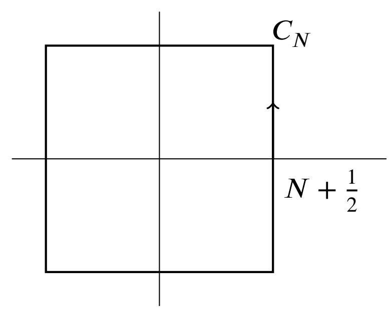
Next observe that if $z = x + iy$ and $y \geq 0$, then $|e^{iz}| = |e^{ix-y}| = |e^{ix}e^{-y}| = e^{-y} \leq 1$. Accordingly, when z lies on the semicircular part of the curve, and $R > 1$, the absolute value of the integrand is bounded above by $1/(R^2-1)^2$. The length of the semicircle is πR, so the integral over the semicircle has absolute value bounded above by $\pi R/(R^2-1)^2$, a quantity that evidently tends to 0 when R tends to infinity.

The upshot is that

$$\frac{\pi}{e} = \int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+1)^2}\,dx = \int_0^\infty \frac{e^{ix}+e^{-ix}}{(x^2+1)^2}\,dx = \int_0^\infty \frac{2\cos(x)}{(x^2+1)^2}\,dx.$$

Dividing by 2 produces the required result.

2. Show that $\displaystyle\lim_{N\to\infty}\int_{C_N} \frac{\pi}{z^2\sin(\pi z)}\,dz = 0$ (where N runs through the natural numbers). Then evaluate the integral by using the residue theorem. Deduce that



$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

Solution. Observe that

$$|\sin(\pi z)| = \left|\frac{e^{i\pi x}e^{-\pi y} - e^{-i\pi x}e^{\pi y}}{2i}\right| = \frac{1}{2}\left|e^{\pi y} - e^{2\pi i x}e^{-\pi y}\right|. \qquad (*)$$

When z lies on either of the two vertical edges of the square, where $x = \pm(N+\frac{1}{2})$, the value of $e^{2\pi i x}$ is -1, so $|\sin(\pi z)|$ reduces to $\cosh(\pi y)$. The Maclaurin series of $\cosh(u)$ is

$1 + \frac{1}{2!}u^2 + \frac{1}{4!}u^4 + \cdots$ with positive coefficients and only even powers, so $\cosh(u) \geq 1$ for every value of u. Thus $1/|\sin(\pi z)| \leq 1$ on the vertical edges of the square. Moreover, applying the triangle inequality to (*) shows that $|\sin(\pi z)| \geq \frac{1}{2}|e^{\pi y} - e^{-\pi y}| = |\sinh(\pi y)|$. The Maclaurin series for $\sinh(u)$ is $u + \frac{1}{3!}u^3 + \cdots$ with all plus signs, so $|\sinh(u)| \geq |u|$ for every value of u. Consequently, if z lies on either of the horizontal edges of the square, then $|\sin(\pi z)| \geq \pi(N + \frac{1}{2}) > 1$, so again $1/|\sin(\pi z)| \leq 1$.

The circle of radius N centered at 0 is inside the square, so $|z^2| > N^2$ when z lies on C_N. Therefore the absolute value of the integral over C_N is no greater than the product of the upper bound π/N^2 for the integrand times the length $4(2N + 1)$ of the path. Evidently $4(2N + 1)\pi/N^2 \to 0$ when $N \to \infty$, so the integral over C_N tends to 0 when N tends to infinity.

There are simple poles inside C_N when z is $\pm 1, \pm 2, \ldots, \pm N$, and there is a triple pole when $z = 0$. The residue at $\pm n$ when $n \neq 0$ is

$$\left.\frac{\pi/z^2}{\frac{d}{dz}\sin(\pi z)}\right|_{z=\pm n}, \qquad \text{or} \qquad \frac{\pi/n^2}{\pi\cos(\pi n)}, \qquad \text{or} \qquad \frac{(-1)^n}{n^2}.$$

One way to compute the residue at 0 is to expand the integrand in a series as follows:

$$\frac{\pi}{z^2\sin(\pi z)} = \frac{\pi}{z^2(\pi z - \frac{1}{3!}\pi^3 z^3 + \cdots)} = \frac{1}{z^3(1 - \frac{1}{3!}\pi^2 z^2 + \cdots)} = \frac{1}{z^3}\left(1 + \frac{1}{3!}\pi^2 z^2 + \cdots\right),$$

the last step following by the geometric-series trick. Therefore the residue at 0, which is the coefficient of $1/z$ in the Laurent series, equals $\pi^2/6$.

Putting all the parts together by the residue theorem shows that

$$0 = \lim_{N\to\infty}\int_{C_N} = 2\pi i \lim_{N\to\infty}\left(\frac{\pi^2}{6} + 2\sum_{n=1}^{N}\frac{(-1)^n}{n^2}\right), \qquad \text{or} \qquad \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$