Quiz

1. Find the radius of convergence of the power series
$$\sum_{n=1}^{\infty} \frac{\log(n)}{n^2 + i^n}(z-2)^n.$$

Solution. Method 1: Apply the root test. The limit that needs to be computed is
$$\lim_{n\to\infty} \left| \frac{\log(n)}{n^2 + i^n}(z-2)^n \right|^{1/n}, \qquad \text{or} \qquad |z-2| \cdot \lim_{n\to\infty} \left| \frac{\log(n)}{n^2 + i^n} \right|^{1/n}.$$
In computing the limit as $n \to \infty$, there is no harm in supposing that n is large, say $n \geq 3$. The triangle inequality then implies that
$$\frac{1}{2}n^2 < n^2 - 1 \leq |n^2 + i^n| \leq n^2 + 1 < 2n^2.$$
Therefore
$$\frac{1}{2^{1/n}} \left(n^{1/n}\right)^2 < |n^2 + i^n|^{1/n} < 2^{1/n} \left(n^{1/n}\right)^2.$$
Since $2^{1/n} \to 1$ when $n \to \infty$, and $n^{1/n} \to 1$ when $n \to \infty$, the squeeze theorem implies that $|n^2 + i^n|^{1/n} \to 1$ when $n \to \infty$. The same assumption on n implies that $1 < \log(n) < n$, so the squeeze theorem similarly implies that $(\log(n))^{1/n} \to 1$ when $n \to \infty$.

The upshot is that the original limit equals $|z-2|$. The root test says that the series converges when $|z-2| < 1$ and diverges when $|z-2| > 1$. Thus the radius of convergence is equal to 1: the series converges in the disk where $|z-2| < 1$ but converges in no larger disk centered at 2.

Method 2: Apply the ratio test. The limit that needs to be computed is
$$\lim_{n\to\infty} \left| \frac{\log(n+1)(z-2)^{n+1}}{(n+1)^2 + i^{n+1}} \Big/ \frac{\log(n)}{n^2 + i^n}(z-2)^n \right|,$$
which reduces to
$$|z-2| \cdot \lim_{n\to\infty} \frac{\log(n+1)}{\log(n)} \left| \frac{n^2 + i^n}{(n+1)^2 + i^{n+1}} \right|.$$
Dividing numerator and denominator by n^2 shows that
$$\lim_{n\to\infty} \frac{n^2 + i^n}{(n+1)^2 + i^{n+1}} = \lim_{n\to\infty} \frac{1 + \frac{i^n}{n^2}}{\left(1 + \frac{1}{n}\right)^2 + \frac{i^{n+1}}{n^2}} = 1.$$
The fundamental property of real logarithms shows that
$$\frac{\log(n+1)}{\log(n)} = \frac{\log(n) + \log\left(1 + \frac{1}{n}\right)}{\log(n)} = 1 + \frac{\log\left(1 + \frac{1}{n}\right)}{\log(n)} \to 1 \quad \text{when } n \to \infty.$$

(Alternatively, you could apply l'Hôpital's rule here.) Therefore the original limit reduces to $|z-2|$, so the series converges when $|z-2|<1$ and diverges when $|z-2|>1$, the same conclusion as before.

2. Find the radius of convergence of the Taylor series of $\tan(z)$ with center at the point $1+i$.

Solution. By the second corollary on page 146 of the textbook, the radius of convergence is the distance from the center point $1+i$ to the nearest singularity of the function $\tan(z)$, which evidently is the point $\pi/2$. So the radius of convergence is $|(\pi/2)-(1+i)|$, or

$$\sqrt{\left(\frac{\pi}{2}-1\right)^2+1}, \qquad \text{or} \qquad \sqrt{\frac{\pi^2}{4}-\pi+2}.$$

Remark. You really do not want to try to compute this Taylor series explicitly! Even the Maclaurin series (center at the origin) of $\tan(z)$ is complicated. Here is how that Maclaurin series is written in formula 4.19.3 of the Digital Library of Mathematical Functions:

$$\tan(z) = z + \frac{z^3}{3} + \frac{2}{15}z^5 + \frac{17}{315}z^7 + \cdots + \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}z^{2n-1} + \cdots,$$

where B_{2n} denotes a so-called Bernoulli number. The Bernoulli numbers, which are named for Jacob Bernoulli (1655–1705), appear in Exercise 4 on page 154 of the textbook, where they are defined via a series expansion for the cotangent function. (Compare formula 4.19.6 in the Digital Library of Mathematical Functions.) But usually the Bernoulli numbers are defined by the following series expansion:

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n.$$

Notice that the function on the left-hand side has a removable singularity at the origin (in the terminology that we just learned).

Values of Bernoulli numbers are available in Maple through the command `bernoulli` and in Mathematica through the command `BernoulliB`. Or you can ask WolframAlpha to tell you the "fourth Bernoulli number" (for example).

The Bernoulli numbers have numerous applications in number theory and in analysis. For instance, you may have encountered the following formulas for special infinite series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

The general case can be formulated in terms of Bernoulli numbers:

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{-(2\pi i)^{2k} B_{2k}}{2(2k)!}, \qquad \text{where } k \text{ is a natural number.}$$