Math 409-502

Harold P. Boas
boas@tamu.edu

Results of the third examination

Good job!

Here are the scores:

95	94	94	92	92	91	90	90	89	88	88
85	85	80	79	78	78	77	76	75	66	34

Reminder: the comprehensive final examination will be held in this room on Tuesday, December 14, from 8:00–10:00 AM.

Things can go wrong in the limit

Continuity. On the interval $[0,1]$, let $f_n(x) = x^{1/n}$.

Then $\lim_{n\to\infty} f_n(x) = \begin{cases} 1, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$

so the limit of these continuous functions is discontinuous.

Derivatives. Let $g_n(x) = \frac{1}{n}x^n$. Then $\left(\lim_{n\to\infty} g_n(x)\right)' = 0$ when $0 \leq x \leq 1$, but $\lim_{n\to\infty} g_n'(x) = \lim_{n\to\infty} x^{n-1} = \begin{cases} 0, & \text{if } x \neq 1, \\ 1, & \text{if } x = 1. \end{cases}$

Here (derivative of the limit) $\neq$ (limit of the derivatives).

Integrals. Let $h_n(x) = \begin{cases} 2^{n+1}, & \text{if } \frac{1}{2^{n+1}} \leq x \leq \frac{1}{2^n}, \\ 0, & \text{otherwise.} \end{cases}$

Then $\int_0^1 \lim_{n\to\infty} h_n(x)\,dx = 0$, but $\lim_{n\to\infty} \int_0^1 h_n(x)\,dx = 1$.

Here (integral of the limit) $\neq$ (limit of the integrals).

Uniform convergence

A sequence of functions $\{f_n\}_{n=1}^{\infty}$ converges *uniformly* on an interval to a function f if for every $\epsilon > 0$ there is an N such that $|f_n(x) - f(x)| < \epsilon$ for all x whenever $n > N$.
"Uniform" means that N can be chosen to be independent of x.

Example. Let $f_n(x) = \sin(x + \frac{1}{n})$. Then the sequence of functions $f_n(x)$ converges uniformly to the function $f(x) = \sin(x)$ on the unbounded interval $(-\infty, \infty)$.

For suppose $\epsilon > 0$ is given. By the mean-value theorem, $\sin(x + \frac{1}{n}) - \sin(x) = \frac{1}{n}\cos(c)$ for some c depending on x, so $|f_n(x) - f(x)| \le \frac{1}{n}$. Therefore $|f_n(x) - f(x)| < \epsilon$ when $n > 1/\epsilon$. So we can take $N = 1/\epsilon$, which is independent of x.

Theorems about uniform convergence

1. If a sequence of continuous functions f_n converges uniformly on an interval to a function f, then the limit function f is continuous.

2. If a sequence of differentiable functions f_n converges (uniformly) on an interval to a function f, and if the sequence of derivatives f_n' converges uniformly to a function g, then the limit function f is differentiable, and $f' = g$.

3. If a sequence of integrable functions f_n converges uniformly on an interval $[a, b]$ to a function f, then f is integrable and $\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx$.
(This is a stronger theorem than the one in the book.)

In all three cases, the word "sequence" can be replaced by the word "series" (because convergence of a series means convergence of the sequence of partial sums).

Homework

Read sections 22.1–22.5, pages 305–318.