Math 409 **Advanced Calculus I** Fall 2004

Third Examination

1. (a) i. Define what it means for a function to be uniformly continuous on an interval.
   ii. Give an example of a function and an interval on which the function is uniformly continuous.
   iii. Give an example of a function and an interval on which the function is continuous but not uniformly continuous.

   (b) i. Define what it means for a bounded function to be integrable on a compact interval.
   ii. Give an example of a bounded function and a compact interval on which the function is integrable.
   iii. Give an example of a bounded function and a compact interval on which the function is not integrable.

2. (a) State the mean-value theorem.

   (b) State Taylor's theorem (with remainder term).

3. For each of the following statements, say whether the statement is true or false. To support an answer of "true", you must give an explanation or cite a theorem or supply a proof; to support an answer of "false", you must exhibit a counterexample.

   (a) If a function f is integrable on the closed interval $[a, b]$, then f is differentiable on the open interval (a, b).

   (b) If a function g is defined on the closed interval $[a, b]$ and is differentiable on the open interval (a, b), then g is integrable on the closed interval $[a, b]$.

4. Two applications of l'Hôpital's rule show that

$$\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \lim_{x\to 0} \frac{\sin(x)}{2x} = \lim_{x\to 0} \frac{\cos(x)}{2} = \frac{1}{2}.$$

   (a) State a version of l'Hôpital's rule that could be used to justify the above calculation.

   (b) Prove the version of l'Hôpital's rule that you stated in part (a).