Math 409-502

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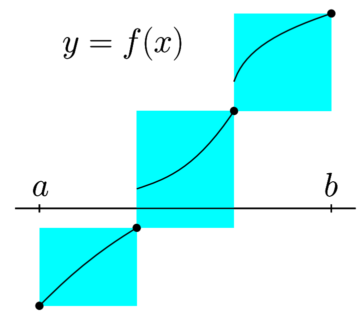
Monotonic functions are integrable

We must show that the difference between the upper sum and the lower sum is small (less than a prescribed ϵ) when the mesh of the partition is small (less than a suitably chosen δ).

Suppose the monotonic function is increasing. On any subinterval of the partition, the difference between the contributions to the upper sum and to the lower sum is at most the mesh size times the difference in function values at the endpoints.
Summing over all subintervals gives a telescoping sum that is at most $(f(b) - f(a)) \times$ mesh.



Take the mesh size to be less than $\epsilon/(1 + f(b) - f(a))$.

Continuous functions are integrable

Suppose a positive ϵ is given.

Because a continuous function on a compact interval $[a, b]$ is *uniformly* continuous, there is a δ such that on every subinterval of width less than δ the maximum and the minimum values of the function differ by less than $\epsilon\ /(b - a)$.

Then for every partition of mesh less than δ, the upper sum and the lower sum differ by less than ϵ.

Homework exercise. Prove that if a function f is defined and bounded on $[a, b]$ and is continuous except at one point, then f is integrable.

Refinements of partitions

A *refinement* of a partition is a new partition obtained by adding some extra division points.

If a partition is refined, the upper sum decreases and the lower sum increases.

Corollary. The upper sum for any partition is at least as big as the lower sum for any other partition.

Proof. Although the upper sum and the lower sum for different partitions are not directly comparable, they can be compared to the upper sum and the lower sum of a common refinement.

Homework

- Read sections 18.3 and 18.4, pages 244–248.
- Do the exercise stated above: Prove that if a function f is defined and bounded on $[a, b]$ and is continuous except at one point, then f is integrable.
- Do exercise 18.4/2 on page 249.