Math 409-502

Harold P. Boas
boas@tamu.edu

Results on the second exam

Maximum 96
Median 81
Minimum 48

Problem 5 on the exam

Determine the radius of convergence of the power series $\sum_{n=0}^{\infty} \left(\frac{1+2^n}{1+n^2}\right) x^n$.

Solution. Here is one of several valid methods.
First observation: since the open interval of convergence $(-R, R)$ is symmetric, we may as well assume that $x > 0$.
Second observation: now the asymptotic comparison test applies, so the new series $\sum_{n=1}^{\infty} \left(\frac{2^n}{n^2}\right) x^n$ converges for the same positive values of x.
By the root test, this new series converges when
$1 > \lim_{n\to\infty} \frac{2x}{n^{2/n}} = 2x$.
So the radius of convergence is $1/2$.

Problem 4(b) on the exam

If a function g has a jump discontinuity at 0, and a function h is continuous at 0, then the product function gh has a jump discontinuity at 0.

True or false?

"Jump discontinuity" means that g has one-sided limits, but $\lim_{x\to 0^-} g(x) \neq \lim_{x\to 0^+} g(x)$.
Since h is continuous, the product function gh has one-sided limits equal to $h(0) \cdot \lim_{x\to 0^-} g(x)$ and $h(0) \cdot \lim_{x\to 0^+} g(x)$. These one-side limits are equal when $h(0) = 0$ and unequal when $h(0) \neq 0$.

So the answer is "false", but the statement is true most of the time (whenever $h(0) \neq 0$).

Problem 4(a) on the exam

If a function f is locally bounded on an interval, then f is bounded on the interval.

True or false?

Theorem 10.4 on page 146 says the statement is true if the interval is *compact*.

On non-compact intervals, however, the statement is false. Example: $1/x$ on the open interval $(0, 1)$.

Problem 3(b) on the exam

Prove from the ϵ-δ definition that the function $1/x$ is continuous at the point 1.

Fix $\epsilon > 0$. We must find $\delta > 0$ such that $\left|\frac{1}{x} - 1\right| < \epsilon$ whenever $|x-1| < \delta$. Now $\left|\frac{1}{x} - 1\right| = \frac{|x-1|}{|x|}$, and the difficulty is that the denominator could be small.

One way to handle the difficulty is to take $\delta = \min(\frac{1}{2}, \frac{\epsilon}{2})$.
If $|x-1| < \delta$, then in particular $|x-1| < \frac{1}{2}$, so $x > \frac{1}{2}$,
whence $\frac{1}{x} < 2$.
Then $\frac{|x-1|}{|x|} \leq 2|x-1| < 2\delta \leq \epsilon$.
Thus we have the required δ.

Homework

Use the ϵ-δ definition of continuity to prove that

1. the function $1/x^2$ is continuous at the point 1;
2. the function $1/x$ is continuous at the point $1/10$.