Math 409-502

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Uniform continuity

Recall that a function f is *continuous* on an interval if for every point a in the interval and for every positive number ϵ there exists a positive number δ such that $|f(x) - f(a)| < \epsilon$ for all x satisfying the inequality $|x - a| < \delta$.
Here the choice of δ may depend on both the point a and the number ϵ.

A function f is *uniformly continuous* on an interval if for every positive number ϵ there exists a positive number δ such that $|f(x) - f(a)| < \epsilon$ for all x and a satisfying the inequality $|x - a| < \delta$.
Here the choice of δ may depend only on the number ϵ.

Examples for uniform continuity

1. On the interval $(0, 1)$, the function x^2 is uniformly continuous.

Proof: Fix $\epsilon > 0$. For all points x and a in the interval $(0, 1)$, $|x^2 - a^2| = |(x - a)(x + a)| \leq 2|x - a|$. Therefore the choice $\delta = \epsilon/2$ works in the definition of uniform continuity.

2. On the interval $(0, 1)$, the function $1/x$ is *not* uniformly continuous.

Proof: Take $\epsilon = 1$. Suppose there were a positive δ such that $\left|\frac{1}{x} - \frac{1}{a}\right| < 1$ for all x and a satisfying the inequality $|x - a| < \delta$. Leaving a arbitrary, set $x = a + \frac{1}{2}\delta$. Then $\left|\frac{1}{x} - \frac{1}{a}\right| = \frac{\frac{1}{2}\delta}{a(a + \frac{1}{2}\delta)}$, which tends to ∞ as $a \to 0^+$. This contradiction shows that the function is not uniformly continuous on the interval $(0, 1)$.

Uniform continuity on compact intervals

Theorem. A continuous function on a *compact* interval is automatically uniformly continuous on the interval.

Proof (different from the proof in the book). Fix $\epsilon > 0$. Suppose there were *no* positive δ that works uniformly on the interval.

Bisect the interval. For at least one half, there is no δ that works uniformly on that half. Bisect again, and iterate.

We get nested compact intervals on each of which there is no δ that works for the given fixed ϵ. The intervals converge to some point a in the original interval.

The function is continuous at a, so there *is* some positive δ for which the values of the function are all within ϵ of each other on the interval $(a - \delta, a + \delta)$.
The contradiction shows that the function must be uniformly continuous after all.

Homework

1. Read section 13.5, pages 190–192.
2. The interval $[0, \infty)$ is not compact. Show nonetheless that the function $\sqrt{x}$ is uniformly continuous on this unbounded interval.
3. The interval $(0, 1)$ is not compact. Determine (with proof) whether $\sin(1/x)$ is uniformly continuous on this open interval.