# Math 409-502

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## The derivative

**Definition.** A function $f$ is *differentiable* at a point $a$ if the limit $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists. The limit, if it exists, is called the *derivative* and is denoted by $f'(a)$.

**Example 1.** If $f(x) = x^2$, then $f'(5) = \lim_{x \to 5} \frac{x^2 - 5^2}{x - 5} = \lim_{x \to 5}(x + 5) = 10$.

**Example 2.** If $f(x) = |x|$, then $f'(0)$ does not exist. Indeed, $\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} 1 = 1$, but $\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} -1 = -1$. So there is a *right-hand derivative* $f'(0^+)$ and there is a *left-hand derivative* $f'(0^-)$, but they are not equal.



## The mean-value theorem

**Theorem.** If $f$ is a continuous function on a compact interval $[a, b]$, and if $f$ is differentiable at all interior points of the interval, then there exists an interior point $c$ for which $f'(c) = \frac{f(b) - f(a)}{b - a}$.

**Example application.** If the derivative of a function is identically equal to zero on an interval, then the function is constant on the interval.

**Proof.** Fix a point $a$ in the interval. Let $b$ be any other point.
By the mean-value theorem, there is a point $c$ for which $f(b) - f(a) = f'(c)(b - a) = 0$. So $f(b) = f(a)$ for every point $b$.

## Proof of the mean-value theorem

Let $g$ be the difference between $f$ and the line joining the points $(a, f(a))$ and $(b, f(b))$: $g(x) = f(x) - \left( f(a) + \dfrac{f(b) - f(a)}{b - a}(x - a) \right)$.

We seek a point $c$ for which $g'(c) = 0$.

The function $g$ is continuous on the compact interval $[a, b]$, so $g$ attains a maximum value and a minimum value. One of these must be attained at an interior point $c$ because $g(a) = 0$ and $g(b) = 0$. We may suppose the maximum is attained at $c$.

Then $g(x) - g(c) \leq 0$ for all $x$, so when $x - c > 0$ we have $\dfrac{g(x) - g(c)}{x - c} \leq 0$, whence $g'(c^+) \leq 0$.

If $x - c < 0$, then $\dfrac{g(x) - g(c)}{x - c} \geq 0$, so $g'(c^-) \geq 0$.

By hypothesis, the derivative $g'(c)$ exists, so the one-sided derivatives are equal. Thus $g'(c) = 0$ as required.



## Homework

1. Read Chapter 14 (pages 196–204) and section 15.1 (pages 210–211).
2. Do Exercise 14.1/3 on page 205.
3. Do Exercise 15.1/4 on page 218.