Math 409-502

Harold P. Boas
boas@tamu.edu

Proving existence of $\lim_{n\to\infty} a_n$

1. If the value of the limit is not known, we can try to use that *a bounded monotonic sequence converges*.
2. If we know or can guess that $a_n \to L$, we can try to estimate the error term $e_n := a_n - L$.

Examples of error term estimation

Powers

$a^n \to 0$ and $na^n \to 0$ when $|a| < 1$;
$a^{1/n} \to 1$ when $0 < a$;
$n^{1/n} \to 1$
Strategy: use the binomial theorem to estimate the error term.

Geometric series $a_n = 1 + r + r^2 + \cdots + r^n$

The error term is $e_n := a_n - \dfrac{1}{1-r} = \dfrac{-r^{n+1}}{1-r}$,
so $e_n \to 0$ when $|r| < 1$ (by the first case above).

Examples continued

Newton's method for approximating $\sqrt{2}$

Here $a_{n+1} = \frac{1}{2}(a_n + \frac{2}{a_n})$, and the error term satisfies a quadratic estimate $|e_{n+1}| \leq |e_n|^2$ if $|e_1| < 0.9$.

Fibonacci fractions $\frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \ldots$

The limit of the sequence is the golden ratio $\dfrac{\sqrt{5}-1}{2}$.
The error term satisfies $|e_{n+1}| \leq \frac{1}{2^n}|e_1|$ if $|e_1| < 0.2$.

A puzzle: $1 = 0$

$$\begin{aligned}
\lim_{n\to\infty} \int_0^1 (n+1)x^n\,dx &= \int_0^1 \lim_{n\to\infty} (n+1)x^n\,dx \\
= \lim_{n\to\infty} \Big[x^{n+1}\Big]_0^1 &= \int_0^1 0\,dx \\
= 1 &= 0
\end{aligned}$$

What went wrong?

The limit of the integral is not necessarily equal to the integral of the limit!

One of our goals in proving theorems about limits is to avoid mistakes like the one above.

Homework for Wednesday

1. Do Exercise 3.6/1 on page 47
   and Exercise 4.4/1 on page 58.

2. Read section 5.1, pages 61–64.