Examination 1

Please write your solutions on your own paper. These problems should be treated as essay questions to answer in complete sentences.

Students in Section 501 should answer questions 1–6 in Parts A and B.

Students in Section 200 should answer questions 1–3 in Part A and questions 7–9 in Part C.

Part A, for both Section 501 and Section 200

1. Give an example of a set E such that the supremum of E equals 5, but E does not have a maximum.

   Solution. One example is the open interval $(2, 5)$. The supremum (least upper bound) is equal to 5, but this value does not belong to the set, so the set has no maximum.

   There are many other correct examples. The extreme example (containing all other examples) is the half-line $\{ x \in \mathbb{R} : x < 5 \}$, which can be written also as $(-\infty, 5)$. A different sort of example is the sequence $\{ 5 - \frac{1}{n} : n \in \mathbb{N} \}$.

2. a) State the definition of what "$\lim_{n\to\infty} x_n = \infty$" means.

   Solution. For every (large) number M, there is a natural number N such that $x_n \geq M$ whenever $n \geq N$ [the statement is Definition 2.9 in the textbook].

   b) Use the definition to prove that $\lim_{n\to\infty} \frac{n-5}{2} = \infty$.

   Solution. If M is given, take N to be a natural number larger than $2M + 5$. (If you want to be fancy, then you can say that such a number exists by the Archimedean property.) If $n \geq N$, then $n - 5 \geq 2M$, so $\dfrac{n-5}{2} \geq M$, which verifies the inequality required by the preceding definition.

3. If E is the set of positive irrational numbers less than $5/2$, then what is the set of interior points of E? Explain.

   Solution. The set of interior points of E is the empty set. Indeed, a point x is an interior point of E precisely when E contains an open interval around x. But the set of irrational numbers contains no intervals at all, since there are rational numbers in every interval—the rational numbers are dense. A set E that contains no intervals has empty interior.

Part B, for Section 501 only

4. The following items are named after famous Greek, Bohemian, German, and French mathematicians:
   - the Archimedean property,
   - the Bolzano–Weierstrass theorem,
   - the Cauchy criterion for convergence.

   Give a precise statement of *one* of these three items.

   Solution. See Theorems 1.11, 2.40, and 2.41 in the textbook.

5. Consider the sequence whose nth term is

$$\frac{(-1)^{5n} + \cos(5n)}{n+5}.$$

   Prove that this sequence converges.

Solution. The difficulty here is that the numerator and the denominator do not converge separately: the numerator oscillates, and the denominator diverges to ∞. A suitable tool for this problem is the sandwich theorem (or squeeze theorem, Theorem 2.20 in the textbook). The numerator is always between -2 and 2, so

$$\frac{-2}{n+5} \le \frac{(-1)^{5n} + \cos(5n)}{n+5} \le \frac{2}{n+5}.$$

Since $2/(n+5) \to 0$, and $-2/(n+5) \to 0$, the squeeze theorem implies that the original sequence converges, indeed converges to the limit 0.

Alternatively, you could argue directly from the definition of limit as follows. If a positive ε is given, and N is taken to be $\lceil 2/\varepsilon \rceil$, then for every value of n greater than N, the following calculation holds:

$$\left| \frac{(-1)^{5n} + \cos(5n)}{n+5} - 0 \right| < \frac{2}{n+5} < \frac{2}{N} \le \varepsilon.$$

Accordingly, the definition of limit shows that the original sequence converges to 0.

Remark This problem is a special case of Exercise 2.8.2, which was a homework problem.

6. True or false: If E is a finite set (that is, a set having only a finite number of elements), then E is necessarily compact.
   If the statement is true, then give a proof; if false, then give a counterexample.

Solution. True. According to the first part of Definition 4.34 in the textbook, a set of real numbers is compact if and only if the set is both closed and bounded. A finite set has no accumulation points and therefore is closed; a finite set has both a maximum and a minimum and therefore is bounded.

Part C, for Section 200 only

7. Here are three famous results encountered in the course so far:
   - the well-ordering property of the natural numbers,
   - Cantor's theorem about uncountability,
   - the Heine–Borel characterization of compact sets.

   Give a precise statement of *one* of these three items.

   Solution. See Theorems 1.12, 2.4, and 4.33 in the textbook.

8. Consider the following recursively defined sequence:

$$x_1 = 2, \qquad \text{and} \qquad x_{n+1} = \sqrt{2 + x_n^2} \quad \text{when } n \geq 1.$$

   Prove that this sequence has no convergent subsequence.

   Solution. Method 1. Since

$$x_{n+1} = \sqrt{2 + x_n^2} > \sqrt{0 + x_n^2} = x_n,$$

   the sequence is strictly increasing. If this increasing sequence is unbounded, then the sequence diverges to ∞, and every subsequence has the same property, so no subsequence converges. On the other hand, if the increasing sequence is bounded, then the sequence converges to the supremum. The goal is to show that this second case does not occur.

Seeking a contradiction, suppose that the sequence is bounded above and converges to the supremum, say L. Squaring the definition of the sequence shows that

$$x_{n+1}^2 = 2 + x_n^2.$$

Passing to the limit, using that limits preserve sums and products, shows that

$$L^2 = 2 + L^2, \qquad \text{or} \qquad 0 = 2.$$

The absurd conclusion demonstrates that the sequence cannot be bounded above after all. This deduction completes the proof.

Method 2. Computing the first few terms of the sequence leads to the conjecture that $x_n = \sqrt{2(n+1)}$ for every natural number n. This conjecture can be proved by induction as follows.

The basis step ($n = 1$) holds because $x_1 = 2 = \sqrt{2 \times 2}$. For the induction step, suppose known for a certain natural number k that $x_k = \sqrt{2(k+1)}$. Then the definition of the sequence implies that

$$x_{k+1} = \sqrt{2 + x_k^2} = \sqrt{2 + 2(k+1)} = \sqrt{2((k+1)+1)}.$$

Accordingly, $x_n = \sqrt{2(n+1)}$ for every natural number n.

Evidently $\sqrt{2(n+1)} \to \infty$, for this sequence is increasing and unbounded (by the Archimedean property, if you want to be fancy). Alternatively, you could say that if a positive number M is given, and if $n > M^2/2$, then

$$\sqrt{2(n+1)} > \sqrt{2n} > M,$$

so the definition of divergence to ∞ is met. Every subsequence diverges to ∞ too, so no subsequence converges.

Examination 1

9. True or false: A set E (a subset of $\mathbb{R}$) is bounded if and only if the closure of E is compact.
   If the statement is true, then give a proof; if false, then give a counterexample.

   Solution. True.

   A set of real numbers is compact if and only if the set is both closed and bounded. The set $\overline{E}$, the closure of E, is a closed set. Therefore $\overline{E}$ is compact if and only if $\overline{E}$ is bounded. Accordingly, the problem reduces to showing that $\overline{E}$ is bounded if and only if E is bounded.

   Suppose that E is a bounded set. Then there exists a positive number M such that every point x in E satisfies the property that $|x| \leq M$. If y is a point of $\overline{E}$ that is not a point of E, then y is a limit of a convergent sequence of points of E. Since limits preserve the order relation $\leq$, such a point y inherits the property that $|y| \leq M$. Therefore $\overline{E}$ is a bounded set (and the same bound that works for E works for $\overline{E}$ too).

   Conversely, if $\overline{E}$ is bounded, then E is bounded, simply because E is a subset of $\overline{E}$. This observation, together with the preceding paragraph, shows that indeed $\overline{E}$ is bounded if and only if E is bounded, so the proof is complete.