Another example of a field that is not ordered

The complex numbers, denoted $\mathbb{C}$, are expressions $a + bi$, where a and b are real numbers, and i is a formal symbol with the property that $i^2 = -1$. The complex numbers form a field, but not an ordered field.

Indeed, if i were positive, then so would be $i \times i$, or -1, contradicting Theorem 1.2.3(10). If instead $-i$ were positive, then so would be $(-i) \times (-i)$, again -1, so the same contradiction arises.

Another example of an ordered field

The rational functions (or rational forms), ratios of polynomials, expressions like

$$\frac{7x^3 - 4x^2 - \sqrt{2}}{x^2 + x - \pi},$$

form a field.

Perhaps surprisingly, there is a way to order this field. Declare a rational function to be positive if the leading coefficient of the numerator and the leading coefficient of the denominator have the same sign (as in the example above).

This field has the remarkable property that the element x, that is, the rational function $x/1$, is not only positive but infinitely large, that is, greater than every natural number! Indeed, to say that $x > n$ is the same as saying that $x - n > 0$, or $\frac{x-n}{1} > 0$, and the indicated rational form is by definition a positive element of the field.

Compare Exercise 6 on page 7 in Section 1.2.

Upper bound

In an ordered field, a nonvoid subset S is *bounded above* if there exists an element k of the field such that $n \leq k$ for every n in S. If S is bounded above, there may or may not be a *least upper bound* (an upper bound b such that every upper bound c has the property that $b \leq c$). The least upper bound is commonly called the *supremum* (plural suprema) of the set, abbreviated $\sup S$.
Example in $\mathbb{Q}$: If $S = \{ x \in \mathbb{Q} : x^2 < 2 \}$, then the supremum of S does not exist (within the declared universe $\mathbb{Q}$).
Example in $\mathbb{R}$: If $S = \{ x \in \mathbb{Q} : x^2 < 2 \}$, then the supremum of S does exist in $\mathbb{R}$ and equals $\sqrt{2}$. In this example, the supremum of S is not itself an element of S.

Completeness

An ordered field is called *complete* if every nonvoid subset that is bounded above has a least upper bound.
Completeness axiom: There exists one (and only one) complete, ordered field: namely, $\mathbb{R}$.