Help

- My office hour is Monday, Wednesday, and Friday afternoons, 2:00–3:00, in Blocker 601L.
- The posted Help Session is Tuesday and Thursday evenings, 7:30–10:00, in Blocker 121.

The greatest-integer function

The proof from last time shows that if x is an arbitrary real number, then there is a unique integer n such that $n \leq x < n + 1$. This integer is denoted either $[x]$ or $\lfloor x \rfloor$ and the function is called either the greatest-integer function or the floor function.
The least integer greater than or equal to x is the ceiling function, denoted $\lceil x \rceil$.

Distance in the real numbers

Absolute value $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$

Geometrically, $|x|$ represents the distance from x to 0, and $|x - y|$ represents the distance from x to y.

Key properties of absolute value:

- $|x| \geq 0$ for every x, and $|x| = 0$ if and only if $x = 0$.
- $|xy| = |x|\,|y|$ for every x and y (multiplicative property).
- $|x + y| \leq |x| + |y|$ (triangle inequality).

Example: Solve $|2x - 3| < 1$.

Solution: Equivalent statement is $-1 < 2x - 3 < 1$, so $1 < x < 2$.

Square root function

Theorem

Every positive real number has a square root. More precisely, if c is a positive real number, then there exists one and only one positive real number x such that $x^2 = c$.

Lemma

If x and y are positive, then $y^2 \geq x^2$ if and only if $y \geq x$.

Proof.

Axioms for ordered fields imply that $y^2 \geq x^2$ if and only if $y^2 - x^2 \geq 0$, equivalently, $(y + x)(y - x) \geq 0$. In an ordered field, inequalities are preserved by multiplying or dividing by a positive quantity, so equivalent is $y - x \geq 0$, that is, $y \geq x$. □

Proof of theorem

Let S denote the set $\{ x \in \mathbb{R} : x \text{ is positive and } x^2 \geq c \}$.
Is S non-empty? Claim: $c + 1$ is an element of S, because $(c+1)^2 = c^2 + 2c + 1 > 2c > c$. So the set S is nonempty, and S is bounded below by 0. Therefore S has a greatest lower bound, say g, by the completeness axiom for $\mathbb{R}$.
The goal is to show that $g^2 = c$. The plan is to show that a contradiction arises if $g^2 < c$, and a different contradiction arises if $g^2 > c$. The law of trichotomy (axiom O_3 on page 4) then yields the goal.
To be continued ....

Assignment to hand in next time

- Exercise 2 on page 14 in Section 1.4.
- Exercise 3 on page 20 in Section 2.4.