Bolzano–Weierstrass theorem

Theorem

Every bounded sequence of real numbers has a convergent subsequence.

Proof by repeated bisection.

By hypothesis, all terms of the sequence (s_n) lie in some interval $[a, b]$. Bisect the interval. The sequence is frequently in either the right half or the left half (or both). Pick an appropriate half, call it $[a_1, b_1]$. Let the first term of the subsequence, s_{n_1}, be the first term of the whole sequence that lies in $[a_1, b_1]$.

Iterate. Bisect $[a_1, b_1]$ and pick a half, $[a_2, b_2]$, that contains infinitely many terms of the original sequence. Let s_{n_2} be the first term of the original sequence that lies in $[a_2, b_2]$ and for which the index n_2 is greater than n_1. And so on.

□

Proof continued

Why does the subsequence converge?
The intervals $[a_n, b_n]$ are nested: namely, the left-hand endpoints a_n are weakly increasing, and the right-hand endpoints b_n are a weakly decreasing sequence. These two monotonic sequences are both bounded (namely, they are inside the original interval $[a, b]$), so a_n converges to something and b_n converges to something.
Observe that $b_n - a_n = (b - a)/2^n$. Therefore (by the squeeze theorem, for instance), the limits of the left-hand endpoints a_n and the right-hand endpoints b_n must be equal.
By construction $a_{n_k} \leq s_{n_k} \leq b_{n_k}$ for each k. By the squeeze theorem, the subsequence s_{n_k} also converges to the same limit as the endpoints of the constructed intervals.

Cantor's nested-interval theorem

If $[a_n, b_n]$ is a sequence of nested closed intervals, then there is a point contained in all of the intervals, that is, $\bigcap_{n=1}^{\infty}[a_n, b_n] \neq \varnothing$. Moreover, if $b_n - a_n \to 0$, then there is exactly one point in the intersection.

Remark: It is important for the intervals to be *closed*.
Example: $\bigcap_n (0, 1/n) = \varnothing$.

Cauchy sequences

A sequence (x_n) of real numbers converges if and only if for every positive ε, there exists N such that $|x_n - x_m| < \varepsilon$ whenever $n \geq N$ and $m \geq N$.