Compactness

A set E is called *sequentially compact* when every sequence of elements of E has a subsequence that converges to a point of E.

Non-example. $E = (0, 1)$ is not sequentially compact, because the sequence $((n-1)/n)$ converges to 1, so every subsequence converges to 1, and the point 1 is not an element of the set E.
Second non-example. $E = \mathbb{R} \setminus \mathbb{Q}$ (the irrational numbers). The sequence $(\sqrt{2}/n)$ converges to 0, which is not an element of the set E.
Alternatively, the sequence $(n\sqrt{2})$ has no convergent subsequence at all, hence we have a second counterexample to compactness of $\mathbb{R} \setminus \mathbb{Q}$.

Example. The closed interval $[0, 1]$ is sequentially compact because every sequence in the set has a convergent subsequence (by Bolzano–Weierstrass) and the limit is in the set because the set is closed.

Bolzano–Weierstrass theorem revisited

Theorem

A subset E of $\mathbb{R}$ is sequentially compact if and only if E is simultaneously closed and bounded.

Heine–Borel covering property

A set E may or may not have the following property: For every collection of open sets whose union contains E, there is some finite subcollection of those sets whose union contains E.
"Every open cover has a finite subcover."

Non-example. $E = (0, 1)$.
$E = \bigcup_{n=2}^{\infty}(1/n, 1)$. Here we have a collection of infinitely many open intervals whose union equals E, but no finite number of these intervals covers E.

Non-example. $E = \mathbb{R}^+ \cup \{0\}$ (the nonnegative real numbers).
The open intervals $(-1/n, n)$ as n runs through the positive integers form a covering of E, but no finite subcollection will do.

Example. $E = [0, 1]$.

Characterizations of compactness in $\mathbb{R}$

Theorem

The following properties of a subset E of $\mathbb{R}$ are equivalent.

1. *E is simultaneously closed and bounded.*
2. *E is sequentially compact.*
3. *E is compact (that is, E satisfies the Heine–Borel covering property).*

Exercise

For each of the following, find an example:

1. A closed set that is not equal to the closure of its interior.
2. An open set that is not equal to the interior of its closure.
3. An infinite compact set whose interior is empty.
4. An open set E and a closed set F such that $E \cup F$ is compact.
5. An open set E and a closed set F such that $E \cap F$ is compact.