# Continuous functions

Suppose $c \in E \subseteq \mathbb{R}$, and $f : E \to \mathbb{R}$ is a function. Then $f$ is *continuous* at the point $c$ when any of the following three equivalent conditions holds.

1. For every sequence $(x_n)$ in $E$ that converges to $c$, the image sequence $(f(x_n))$ converges to $f(c)$.
2. For every positive $\varepsilon$, there exists a positive $\delta$ such that $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$ and $x \in E$; in symbols, $\forall\varepsilon > 0\ \exists\delta > 0$ such that $\forall x \in E\ \ |x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$.
   Negation: $\exists\varepsilon > 0$ such that $\forall\delta > 0\ \exists x \in E$ for which $|x - c| < \delta$ but $|f(x) - f(c)| \geq \varepsilon$.
3. For every neighborhood $V$ of $f(c)$, the inverse image $f^{-1}(V)$, that is, $\{x \in E : f(x) \in V\}$, is a neighborhood of $c$.

# Example

Suppose $E = \mathbb{R}^+$, $f(x) = 1/x$, and $c = 2$.

Why is $f$ continuous at 2?
If $(x_n)$ is an arbitrary sequence of positive real numbers, and if $x_n \to 2$, then by known properties of limits of sequences,

$$\frac{1}{x_n} \to \frac{1}{2},$$

so $f(x_n) \to f(c)$. Thus the first definition of continuity is met.

# Example continued

Check continuity of $1/x$ at 2 using the second definition.
Suppose $\varepsilon$ is an arbitrary positive number.
Goal: find a positive $\delta$ such that

$$\left|\frac{1}{x} - \frac{1}{2}\right| < \varepsilon \qquad \text{when } |x-2| < \delta \quad \text{and } x > 0.$$

Side calculation: $\dfrac{1}{x} - \dfrac{1}{2} = \dfrac{2-x}{2x}$. One way to guarantee that the fraction is close to zero is to make the numerator close to zero and the denominator stay away from zero.
If $|x-2| < 1$ (for example), then $-1 < x-2 < 1$, so in particular, $1 < x$, hence $\left|\frac{2-x}{2x}\right| \le \frac{|x-2|}{2}$.
Now take $\delta$ to be $\min\{1, \varepsilon\}$. If $|x-2| < \delta$, then

$$\left|\frac{1}{x} - \frac{1}{2}\right| \le \frac{|x-2|}{2} \le \frac{\varepsilon}{2} < \varepsilon.$$

# Two big theorems

**Theorem (Intermediate-value theorem)**

*If $I$ is an interval, and $f\colon I \to \mathbb{R}$ is continuous at every point of $I$, then the image $f(I)$ is an interval.*

**Theorem (Extreme-value theorem)**

*If $K$ is a compact subset of $\mathbb{R}$, and $f\colon K \to \mathbb{R}$ is continuous at every point of $K$, then $f$ attains a maximum value on $K$ (and also attains a minimum value).*