Coming attractions

- Limits of functions
- Continuous functions
- Theorems about continuous functions on intervals
- Differentiable functions
- Theorems about differentiable functions on intervals
- Riemann integration

Limits of functions

Suppose $f\colon E \to \mathbb{R}$, and c is a limit point of the domain E (not necessarily a point of E); that is, there is a sequence (x_n) of points of $E \setminus \{c\}$ that converges to c.

Definition

To say that $\lim_{x\to c} f(x) = L$ means

- for every sequence (x_n) in $E \setminus \{c\}$, if $x_n \to c$ then $f(x_n) \to L$; equivalently,
- for every positive ε there exists a positive δ such that if $x \in E \setminus \{c\}$ and $|x - c| < \delta$ then $|f(x) - L| < \varepsilon$.

Often E is an interval (open or closed) and c is either an interior point of the interval or an endpoint of the interval.

Example

Suppose E is the open interval $(0, 1)$ and $f\colon E \to \mathbb{R}$ is defined as follows: $f(x) = \sin(1/x)$ for x in E.

What can you say about $\lim_{x \to 0} f(x)$?
Since $f(x) = 1$ when $x = 2/\pi$ and $2/(5\pi)$ and $2/(9\pi)$ and so on, and this sequence $(2/((1 + 4n)\pi))$ has limit 0; but $f(2/(3\pi)) = -1$ and generally $f(2/((3 + 4n)\pi)) = -1$; so the function cannot have a limit at 0, for there are different limits along different sequences.

Another example of failure

$E = (0, 1)$, $f(x) = 1/x$.

$\lim_{x \to 0} f(x)$ fails to exist because $f(x_n)$ is unbounded for every sequence (x_n) that approaches 0.

A fancier example

Suppose E is the set of positive rational numbers, and $f\colon E \to \mathbb{R}$ is defined as follows: $f(m/n) = m/n^2$ when m and n are positive integers with no common factor.

What can you say about $\lim_{x \to 1} f(x)$?

If $x_n \to 1$ but $x_n \neq 1$, then the denominator of x_n is growing without bound, and $f(x_n)$ is approximately the reciprocal of the denominator of x_n, so $f(x_n) \to 0$ for every such sequence.
So $\lim_{x \to 1} f(x) = 0$ even though $f(1) = 1$.

How about $\lim_{x \to \pi} f(x)$?
Limit is zero for essentially the same reason.

Continuous functions

Suppose $f\colon E \to \mathbb{R}$, and c is a point of the domain E.

Definition

To say that f is continuous at c means

- for every sequence (x_n) in E, if $x_n \to c$ then $f(x_n) \to f(c)$; equivalently,
- $\lim_{x \to c} f(x)$ exists and equals $f(c)$; equivalently,
- for every positive ε there exists a positive δ such that if $x \in E$ and $|x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$.