Properties of continuous functions

Are the continuous functions $f\colon \mathbb{R} \to \mathbb{R}$

- a field?
  No: $\sin(x)$ is continuous, but the reciprocal $1/\sin(x)$ is not continuous.
- a vector space?
  Yes.
- closed under taking the maximum?
  In other words, if $f(x)$ and $g(x)$ are continuous functions, is $\max\{f(x), g(x)\}$ continuous too?
  Yes.

Proof for continuity of maximum at a point c

Suppose ε is a prescribed positive number.
By hypothesis, there is a positive δ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$ and $|g(x) - g(c)| < \varepsilon$, or, equivalently,

$$-\varepsilon < f(x) - f(c) < \varepsilon \quad \text{and} \quad -\varepsilon < g(x) - g(c) < \varepsilon$$

or

$$f(c) - \varepsilon < f(x) < f(c) + \varepsilon \quad \text{and} \quad g(c) - \varepsilon < g(x) < g(c) + \varepsilon.$$

But $f(c) + \varepsilon \leq \max\{f(c), g(c)\} + \varepsilon$ and
$g(c) + \varepsilon \leq \max\{f(c), g(c)\} + \varepsilon$.
Combining these inequalities shows that
$\max\{f(x), g(x)\} \leq \max\{f(c), g(c)\} + \varepsilon$.
A similar argument shows that
$\max\{f(c), g(c)\} - \varepsilon \leq \max\{f(x), g(x)\}$. So we are done.

Nutty ionic exercise

Continuity of f at c means:
$\forall\varepsilon > 0 \; \exists\delta > 0$ such that $|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$.

What do each of the following mangled properties mean?

1. $\exists\varepsilon > 0 \; \forall\delta > 0 \;\; |x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$.
2. $\forall\varepsilon > 0 \; \forall\delta > 0 \;\; |x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$.
3. $\exists\varepsilon > 0 \; \exists\delta > 0$ such that $|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$.
4. $\forall\varepsilon < 0 \; \exists\delta < 0$ such that $|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$.