Announcement

Math Club Meeting
Tuesday, April 18th, 2017
Blocker 220
7:00–8:00 PM

Agenda:

- officer elections
- food
- a talk by Dr. Florent Baudier

Three ways of looking at the derivative

Suppose $f\colon I \to \mathbb{R}$ is a function whose domain is an open interval I, and c is a point in I.
There are three equivalent ways to define the derivative $f'(c)$:

1. using limits
2. leveraging the notion of continuity
3. formalizing the geometric picture

(Nothing essential changes if I is a closed interval, and c is an endpoint. The concept then is a *one-sided* derivative.)

Definition of the derivative using limits

The function f is differentiable at the point c if and only if

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \qquad \text{exists,}$$

in which case the value of the limit is called the derivative, denoted by $f'(c)$.
Replacing x by $c + h$ yields the equivalent formulation that

$$f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$

if this limit exists.

Definition of the derivative using continuity

The function f is differentiable at c if and only if there exists a function A, continuous at c, such that $f(x) = A(x)(x - c) + f(c)$. And $f'(c) = A(c)$.

The only thing $A(x)$ can be when $x \neq c$ is the fraction

$$\frac{f(x) - f(c)}{x - c}.$$

To say that A is continuous at c means precisely that this fraction has a limit when $x \to c$, and the value of the limit is $A(c)$.

Definition of the derivative using geometry

The function f has a tangent line at c when there is a "best linear approximation," that is, a linear function T such that

$$\lim_{h \to 0} \frac{f(c+h) - f(c) - T(h)}{h} = 0.$$

What $T(h)$ has to be is $f'(c)h$.

(In higher dimensions, the right way to think about the derivative is not as a number but as a linear transformation.)

Confirming some prior knowledge

Example

If P is a polynomial, then P is differentiable at every real number c.

Proof.

From algebra, the difference $P(x) - P(c)$ is divisible by $(x - c)$: namely, there is a polyomial $Q(x)$ such that $P(x) - P(c) = (x - c)Q(x)$. Since polynomials are continuous functions, the second definition of differentiability shows that P is differentiable at c, and $P'(c) = Q(c)$. $\square$

Some fancier examples

Are the following functions differentiable at 0?

1. $f(x) = x|x|$
2. $g(x) = \begin{cases} x \sin(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$
3. $h(x) = \begin{cases} x^2 \sin(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$
4. $k(x) = \begin{cases} x^2, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$

Answer: yes for f, h, and k, but no for g.