Reminder

The comprehensive final examination takes place on Thursday, May 4, from 12:30 to 2:30 in the afternoon, here in this room.

Warm-up on derivatives

Example (#8 on page 134)

Let $f\colon [1,3] \to \mathbb{R}$ be a continuous function that is differentiable on $(1,3)$ with derivative $f'(x) = [f(x)]^2 + 4$ for all $x \in (1,3)$. True or false (explain): $f(3) - f(1) = 5$.

Mean-value theorem implies the existence of a point c for which

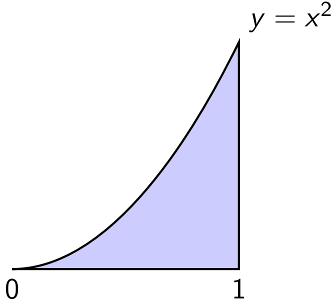
$$\frac{5}{2} = \frac{f(3) - f(1)}{3 - 1} = f'(c) \geq 4$$

which is a contradiction.

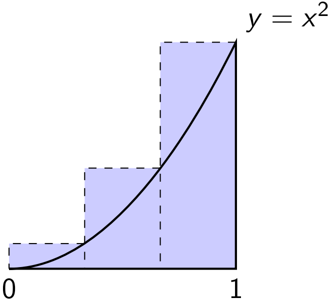
Warm-up on area

Example

How could pre-Newtonian mathematicians find the area under a parabola?



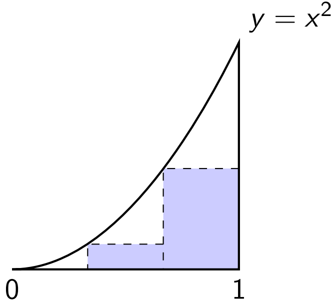
An upper bound for the area



For n rectangles of equal width, the upper bound for the area equals

$$\sum_{i=1}^{n} \text{width} \times \text{height} = \sum_{i=1}^{n} \frac{1}{n} \times \left(\frac{i}{n}\right)^2$$

A lower bound for the area



For n rectangles of equal width, the lower bound for the area equals

$$\sum_{i=0}^{n-1} \text{width} \times \text{height} = \sum_{i=0}^{n-1} \frac{1}{n} \times \left(\frac{i}{n}\right)^2$$

The separation between upper and lower bounds

$$\text{upper bound} - \text{lower bound} = \frac{1}{n}\left[\left(\frac{n}{n}\right)^2 - \left(\frac{0}{n}\right)^2\right] = \frac{1}{n}$$

So the area is approximated within ε by either the upper bound or the lower bound as soon as $n > 1/\varepsilon$.

Exact computation of the area

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \left(\frac{i}{n}\right)^2 = \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2$$

Mathematical induction shows that $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$, so

$$\text{Area} = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{3}.$$

Example

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Interval $[0, 1]$:

Here the upper bound is always 1 and the lower bound is always 0, so the approximation scheme fails in this example.