Reminders

- The last class meeting is today.
- Please fill out the course evaluation form at `http://www.math.tamu.edu/`.
- The comprehensive final examination takes place on Thursday, May 4, from 12:30 to 2:30 in the afternoon, here in this room.

  As usual, please bring your own paper to the exam.
- Next week, I will hold my usual office hour on Monday and Wednesday afternoons from 2:00 to 3:00.

Cauchy's theorem on the integral from last time

A function $f\colon [a,b] \to \mathbb{R}$ is *Riemann integrable* if there is a real number I (the value of the integral) such that for every positive tolerance ε, there exists a positive δ with the following property: for every partition of $[a,b]$ into subintervals $[x_0,x_1]$, $[x_1,x_2]$, ..., $[x_{n-1},x_n]$ (where $x_0 = a$ and $x_n = b$), each of length less than δ, and for every choice of t_k in the kth subinterval $[x_{k-1},x_k]$, the difference $I - \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1})$ has absolute value less than ε.

Theorem (Cauchy)

If f is a continuous function on a compact interval $[a,b]$, then f is Riemann integrable on $[a,b]$.

Idea of the proof of Cauchy's theorem on the integral

By the magic theorem, f is uniformly continuous on $[a, b]$, so there is a δ with the property that the values of f change by less than $\varepsilon/(b-a)$ on every interval of length less than δ. Now

$$\sum_{k=1}^{n}\left(\min_{x_{k-1}\le x\le x_k} f(x)\right)(x_k - x_{k-1}) \le \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) \le \sum_{k=1}^{n}\left(\max_{x_{k-1}\le x\le x_k} f(x)\right)(x_k - x_{k-1})$$

and the upper and lower bounds differ by less than ε if each subinterval has width less than δ. Let I be the supremum of the lower bounds over all partitions (= infimum of upper bounds over all partitions).

One part of the fundamental theorem of calculus

Suppose f is a continuous function, and F is a differentiable function such that $F'(x) = f(x)$ for all x; that is, F is an antiderivative (or primitive) of f. Then $\int_a^b f(t)\,dt = F(b) - F(a)$. Why? Write the right-hand side as a telescoping sum:

$$F(b) - F(a) = F(x_n) - F(x_{n-1}) + F(x_{n-1}) + \cdots + F(x_1) - F(x_0).$$

By the mean-value theorem, this sum equals

$$\sum_{k=1}^{n} F'(t_k)(x_k - x_{k-1}) = \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1})$$

for some choice of t_k between x_k and x_{k-1}. Pass to the limit.

The other part of the fundamental theorem of calculus

If f is continuous, and $F(x) = \int_a^x f(t)\,dt$, then F is differentiable, and $F'(x) = f(x)$.

Example

$$\frac{d}{dx}\int_0^{x^2} \sin(t)\,dt = \frac{d}{dx}\int_0^{u(x)} \sin(t)\,dt$$

Chain rule: $2x\sin(x^2)$.

Example

If $G(x) = \int_{x^3}^{\sin(x)} \sqrt{1+t^2}\, dt$, find $G'(x)$.

Solution.
$G(x) = F(\sin(x)) - F(x^3)$,
so $G'(x) = F'(\sin(x))\cos(x) - F'(x^3)3x^2 =$
$\sqrt{1+\sin^2(x)}\cos(x) - \sqrt{1+x^6}\, 3x^2$ □