Examination 2

Instructions: Please write your solutions on your own paper. These problems should be treated as essay questions to answer in complete sentences.

1. For each part, give an example of a subset of $\mathbb{R}$ satisfying the specified property.
   a) An unbounded open set whose complement is unbounded too.
   b) A non-empty compact set having empty interior.

2. Suppose $f: (0,1) \to \mathbb{R}$ is defined as follows:
$$f(x) = \sqrt{x}, \qquad 0 < x < 1.$$
(You know from Section 2.8 that every positive real number has a unique positive square root, so f is well defined.) Prove the unsurprising fact that
$$\lim_{x \to 0} f(x) = 0.$$

3. Give an example of a function $f: (0,1) \to \mathbb{R}$ that is continuous at every point of the interval $(0,1)$ but is not uniformly continuous on this interval. Explain why your example works.

4. State **one** of the following theorems (your choice):
   a) the intermediate-value theorem, or
   b) the extreme-value theorem, or
   c) the Heine–Borel covering theorem.

5. Suppose $f: (0,1) \to \mathbb{R}$, and let S denote the set $\{\, x \in (0,1) : f \text{ is continuous at } x \,\}$. Must S be an open set? Supply a proof or a counterexample, as appropriate.

6. Suppose $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are two continuous functions. Prove that if $f(r) = g(r)$ for every rational number r, then $f(x) = g(x)$ for every real number x.

Extra Credit Problem. A theorem about inverse functions says that if I and J are intervals in $\mathbb{R}$, and a function f is a continuous bijection from I onto J, then the inverse function f^{-1} is automatically continuous on J.

Your task is to construct an example of two subsets A and B of $\mathbb{R}$ and a bijective continuous function f from A onto B such that f^{-1} is discontinuous at some point of B. (In view of the theorem, your sets A and B cannot both be intervals.)