Examination 1

Part A: Sentence Completion

Your answer to each of problems 1–3 should be a complete sentence that starts as indicated.

1. The statement "$\lim_{n\to\infty} x_n = L$" means that for every positive real number ε, there exists ....
2. The set of real numbers is the only ordered field that additionally ....
3. The Archimedean property states that ....

Part B: Examples

Your task in problems 4–5 is to exhibit a concrete example satisfying the indicated property. You should provide a brief explanation of why your example works.

4. Give an example of a set of real numbers that has a supremum but not a maximum.
5. Give an example of a bounded sequence of real numbers that does not converge.

Part Γ: Proof

Your proof should be written in complete sentences, each step being justified. You may invoke theorems from the textbook, in which case you should indicate what the cited theorems say.

6. Suppose $x_1 = 5$, and $x_{n+1} = \dfrac{1 + x_n}{2}$ when $n \geq 1$. Prove that this recursively defined sequence $\{x_n\}_{n=1}^{\infty}$ converges.

Part Δ: Optional Extra Credit Problem

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Suppose

$$y_n = x_n^2 \qquad \text{and} \qquad z_n = \frac{x_n}{x_n^2 + x_n + 1} \qquad \text{when } n \in \mathbb{N}.$$

Prove that if both of the sequences $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ converge, then the sequence $\{x_n\}_{n=1}^{\infty}$ must converge too.