Reminder

The first exam takes place in class on February 16
(a week from today).

Material covered on the exam: Sections 0.3, 1.1–1.4, and 2.1–2.2.

A general question

The real numbers have

1. an algebraic structure (addition and multiplication)
2. an order structure (the relation $\leq$ makes $\mathbb{R}$ into an ordered field)
3. a metric structure ($|x - y|$ represents the distance between x and y)

Is the operation of taking limits compatible with these three structures?

Limits and algebraic operations

Are the following properties true?

$$\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$$
$$\lim_{n\to\infty}(x_n y_n) = \left(\lim_{n\to\infty} x_n\right)\left(\lim_{n\to\infty} y_n\right)$$

Yes, if the two limits on the right-hand side exist (Proposition 2.2.5).

Limits and the order relation

Are the following properties true?

$$(\forall n \;\; x_n < y_n) \implies \lim_{n\to\infty} x_n < \lim_{n\to\infty} y_n$$

$$(\forall n \;\; x_n \leq y_n) \implies \lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$$

If the limits exist, then the second property is true (Lemma 2.2.3), but the first property can fail.

Example

If $x_n = -1/n$ and $y_n = 1/n$, then $x_n < y_n$, but $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$ (both limits equal 0).

Limits and absolute value

Is the following property true?

$$\left|\lim_{n\to\infty} x_n\right| = \lim_{n\to\infty} |x_n|$$

Yes, if the limit on the left-hand side exists (Proposition 2.2.7). But it could happen that the limit on the right-hand side exists even though the limit on the left-hand side does not exist.

Example

If $x_n = (-1)^n$, then $\lim_{n\to\infty} |x_n| = 1$, but $\left|\lim_{n\to\infty} x_n\right|$ does not exist.

Zero is special

Every limit can be reduced to a question about convergence to 0:

$$\lim_{n\to\infty} x_n = L \iff \lim_{n\to\infty} (x_n - L) = 0 \iff \lim_{n\to\infty} |x_n - L| = 0.$$

Some conditions guaranteeing convergence to 0:

- Geometric sequence: if $|c| < 1$, then $\lim_{n\to\infty} c^n = 0$.
- Comparison test: if $\forall n\ (|x_n| \leq a_n)$, and if $\lim_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} x_n = 0$.
  [special case of squeeze theorem]
- Ratio test: if $\lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = L$, and if $L < 1$, then $\lim_{n\to\infty} x_n = 0$.

Assignment due next class

1. Write solutions to Exercise 1.3.5 and Exercise 2.2.5.
2. Read subsection 2.2.4 in the textbook.