Some standard convergence tests for positive series [from last time]

- geometric series
- comparison test
- Cauchy's condensation test for monotonic series [not in book]
- root test
- ratio test

Cauchy's root test [from last time]

Suppose $x_n \geq 0$ for every n. Then the series $\sum_{n=1}^{\infty} x_n$

- converges if $\limsup_{n\to\infty} x_n^{1/n} < 1$
- diverges if $\limsup_{n\to\infty} x_n^{1/n} > 1$.

If $\limsup_{n\to\infty} x_n^{1/n} = 1$, the test gives no information.

Ratio test for series (Jean le Rond d'Alembert, 1717–1783)

Suppose $x_n \geq 0$ for every n. Suppose $\lim_{n\to\infty} \frac{x_{n+1}}{x_n}$ exists and equals L.

1. If $L < 1$, then
   (a) $\lim_{n\to\infty} x_n = 0$ (from the ratio test for *sequences*), and
   (b) the series $\sum_{n=1}^{\infty} x_n$ converges.
2. If $L > 1$, then
   (a) the terms of the sequence $\{x_n\}_{n=1}^{\infty}$ are unbounded, and
   (b) the series $\sum_{n=1}^{\infty} x_n$ diverges.
3. If either $L = 1$ or $\lim_{n\to\infty} \frac{x_{n+1}}{x_n}$ does not exist, then the test gives no information.

Example: the root test is more general than the ratio test

Suppose $x_n = \begin{cases} \frac{1}{2^n}, & \text{when } n \text{ is even,} \\ \frac{1}{3^n}, & \text{when } n \text{ is odd.} \end{cases}$

Then the ratio $\frac{x_{n+1}}{x_n}$ is alternately $\frac{3^n}{2^{n+1}}$ and $\frac{2^n}{3^{n+1}}$, so $\lim_{n\to\infty} \frac{x_{n+1}}{x_n}$ does not exist.

But $\limsup_{n\to\infty} x_n^{1/n} = \frac{1}{2} < 1$, so the series $\sum_{n=1}^{\infty} x_n$ converges by the root test.

Exercises

Determine which of these series converge.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 4^n}{3^n + 5^n}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{n! + n}{n^n}$
3. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{\log(n)}}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2 + 1}{2n^3 + n}$

Assignment due next class

Think warm thoughts about sequences and series.