Series with some positive and some negative terms

Example: the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Although the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ (with all plus signs) diverges, the series with alternating signs converges. Why?

Pair up consecutive terms to write the series as

$\sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n} \right)$ or $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n)}$ and observe that $\frac{1}{(2n-1)(2n)} < \frac{1}{n^2}$. The paired-up series has positive terms and converges by comparison with the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

It is not obvious that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \log(2)$.

Absolute convergence

Theorem

If $\sum_{n=1}^{\infty} |x_n|$ *converges, then* $\sum_{n=1}^{\infty} x_n$ *converges.*

"An absolutely convergent series converges."

Proof.

The goal is to show that the partial sums of the series form a Cauchy sequence. So fix a positive ε. We seek an M such that whenever $n \geq M$ and $m \geq M$, we have $\left|\sum_{k=1}^{n} x_k - \sum_{k=1}^{m} x_k\right| < \varepsilon$, or $\left|\sum_{k=m+1}^{n} x_k\right| < \varepsilon$. By the triangle inequality, $\left|\sum_{k=m+1}^{n} x_k\right| \leq \sum_{k=m+1}^{n} |x_k|$, so the M that works for $\sum_k |x_k|$ also works for $\sum_k x_k$. □

If absolute convergence fails, what tests are available?

Theorem (Alternating series test)

If $\{x_n\}_{n=1}^{\infty}$ *is a* **decreasing** *sequence of positive numbers, and if* $\lim_{n\to\infty} x_n = 0$, *then the series* $\sum_{n=1}^{\infty}(-1)^n x_n$ *converges.*

Theorem (Dirichlet's test)

If $\{x_n\}_{n=1}^{\infty}$ *is a* **decreasing** *sequence of positive numbers, and if* $\lim_{n\to\infty} x_n = 0$, *and if* $\{y_n\}_{n=1}^{\infty}$ *is a sequence that has bounded partial sums, then* $\sum_{n=1}^{\infty} x_n y_n$ *converges.*

[The alternating series test is the special case of Dirichlet's test when $y_n = (-1)^n$.]

Non-obvious example: $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n)}{n}$ converges.

Assignment over Spring Break

Travel safely, and converge absolutely to College Station in the limit as t tends to $3/18$.