Recap: Continuous functions

A function $f\colon S \to \mathbb{R}$ is *continuous at a point c in S* when

- either c is an isolated point of S,
- or c is a cluster point of S, and
  - $\lim_{x \to c} f(x)$ exists, and
  - $\lim_{x \to c} f(x) = f(c)$.

Equivalent statement in terms of sequences:
For every sequence $\{x_n\}_{n=1}^{\infty}$ of points of S such that $\lim_{n \to \infty} x_n = c$, the image sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(c)$.

Equivalent quantified statement:
For every positive ε, there exists a positive δ such that the inequality $|x - c| < \delta$ implies the inequality $|f(x) - f(c)| < \varepsilon$ when x is in S.

Two important properties of the range of a continuous function whose domain is a closed bounded interval

Theorem

Suppose $f\colon [a, b] \to \mathbb{R}$ is continuous. Then

1. *the range of f is a bounded set that contains both its supremum and its infimum, and*
2. *the range of f is an interval*
   (a degenerate interval if f is a constant function).

Conclusion 1 is the *extreme-value theorem* (called the min–max theorem in the book). It says that the function f attains a maximum value and also attains a minimum value.

Conclusion 2 is the *intermediate-value theorem*. It says that if numbers c and d are in the range of f, then every number between c and d is in the range of f.

Proof of the intermediate-value theorem

The main step is to show that if $f(a) < f(b)$, and v is a value between $f(a)$ and $f(b)$, then there is some x between a and b such that $f(x) = v$. Strategy: use the least-upper bound property of $\mathbb{R}$.

Let E be $\{\, x \in [a, b] : f(x) < v \,\}$, and let s denote $\sup E$.
Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points of E having limit equal to s.

The image sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(s)$ because f is continuous.
But $f(x_n) < v$ for every n, so $f(s) \leq v$.

Now $s < b$, and if n is large enough that $s + \frac{1}{n} < b$, then $f(s + \frac{1}{n})$ is defined and $\geq v$. So $f(s) = \lim_{n\to\infty} f(s + \frac{1}{n}) \geq v$.

Since $f(s) \leq v$ and also $f(s) \geq v$, the number s is the one we seek.

Assignment due next class

Write solutions to Exercises 3.3.1 and 3.3.2.