Warm-up Exercise

Prove by induction that the derivative of x^n equals $n\,x^{n-1}$ when n is a positive integer.

Proof.

Induction step: Suppose n is a natural number for which we know that the derivative of x^n is nx^{n-1}. Then view x^{n+1} as the product of x and x^n.

The product rule implies that the derivative of $x \cdot x^n$ equals the sum of the two terms $1 \cdot x^n$ and $x \cdot nx^{n-1}$, which equals $(n+1)x^n$. □

Refresher on the chain rule

If $f(x) = \sin(2x + \tan(x))$,
then $f'(x) = \cos(2x + \tan(x)) \cdot (2 + \sec^2(x))$.

In general, $f \circ g$ means the function whose value at x is $f(g(x))$,
and $(f \circ g)'(x) = f'(g(x))g'(x)$.

Proof of the chain rule for composite functions

Suppose g is differentiable at c, and the range of g is a subset of the domain of f, and f is differentiable at $g(c)$.
Property 2 provides a function F, continuous at $g(c)$, such that

$$\begin{aligned} f(y) &= F(y)(y - g(c)) + f(g(c)), \qquad \text{so} \\ f(g(x)) &= F(g(x))(g(x) - g(c)) + f(g(c)). \end{aligned}$$

There is a function G, continuous at c, such that

$$g(x) - g(c) = G(x)(x - c), \qquad \text{so}$$

$$f(g(x)) = F(g(x))G(x)(x - c) + f(g(c)).$$

Therefore the composite function $f(g(x))$ is differentiable at c, and the derivative at c equals $F(g(c))G(c)$, or $\boxed{f'(g(c))g'(c)}$.

Assignment due next class

Write a solution to Exercise 4.1.10.