What goes up must come down

Theorem (Rolle's theorem)

If $f : [a, b] \to \mathbb{R}$ *is continuous, and* $f(a) = f(b)$*, and the derivative* f' *exists at all points of* (a, b)*, then there is some point* c *(at least one) in the interval* (a, b) *for which* $f'(c) = 0$*.*

Proof.

Apply the extreme-value theorem. If the max and min both occur at the endpoints, then the function is constant, so f' is identically zero.

Otherwise, let c be an interior point where there is an extreme value, WLOG a maximum. The numerator of the fraction $\dfrac{f(x) - f(c)}{x - c}$ is ≤ 0. The denominator is positive when $x > c$ and negative when $x < c$, so the limit of the fraction is both ≤ 0 and ≥ 0, hence $= 0$. And this limit is $f'(c)$. □

Example

Suppose

$$p(x) = x^5 + 3x^4 - 5x^3 - 15x^2 + 4x + 12.$$

Observe that $p(0) = 12$, $p(1) = 0$, and $p(-1) = 0$, so the derivative $5x^4 + 12x^3 - 15x^2 - 30x + 4$ must equal 0 at some point between -1 and 1.

The mean-value theorem

Theorem (mean-value theorem, basic version)

If $f\colon [a,b] \to \mathbb{R}$ *is continuous, and the derivative* f' *exists at all points of* (a,b)*, then there exists a point* c *in* (a,b) *for which*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Assignment due next class

Write a solution to Exercise 4.2.11.