Recap: second-order Taylor theorem

If f'' (the second derivative) exists on some interval containing a and x, then there is some c between a and x for which

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(c)(x-a)^2.$$

Example: $f(x) = \sin(x)$, $a = 0$

$\sin(x) = x + \frac{1}{2}(-\sin(c))x^2$ for some c between 0 and x.
Taylor's formula can be an alternative to l'Hôpital's rule:

$$\lim_{x\to 0}\frac{\sin(x)}{x} = \lim_{x\to 0}\frac{x + \frac{1}{2}(-\sin(c_x))x^2}{x} = 1 + \lim_{x\to 0}\frac{1}{2}(-\sin(c_x))x$$

Since $|-\sin(c_x)| \le 1$, the squeeze theorem implies that the final limit equals 0, so $\lim\limits_{x\to 0}\frac{\sin(x)}{x} = 1$.

Proof of Taylor's formula with second-order remainder

Define two new functions:

$$F(x) = f(x) - f(a) - f'(a)(x-a) \quad \text{and} \quad G(x) = \tfrac{1}{2}(x-a)^2.$$

By Cauchy's mean-value theorem, there exists a point c between a and x for which

$$\begin{aligned} F'(c)(G(x) - G(a)) &= G'(c)(F(x) - F(a)), \\ \text{or} \qquad F'(c)G(x) &= G'(c)F(x). \end{aligned}$$

Divide by $G'(c)$ to see that

$$F(x) = \frac{f'(c) - f'(a)}{c-a} \cdot \frac{1}{2}(x-a)^2 = \frac{1}{2} f''(c_1)(x-a)^2$$

by another application of the mean-value theorem.

The general Taylor formula

If f is $(n+1)$ times differentiable on an open interval containing x and a, then there is a point c between a and x for which

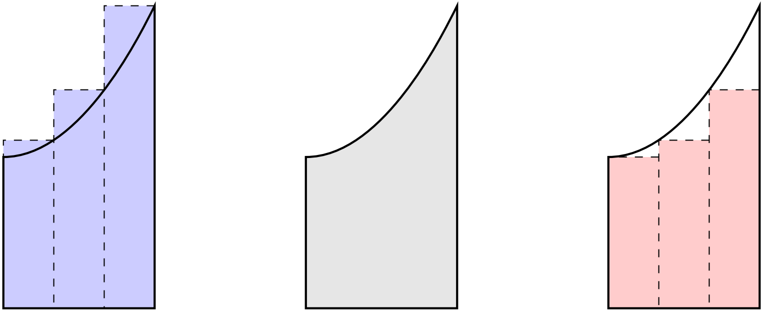
$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n + \frac{1}{(n+1)!}f^{(n+1)}(c)(x-a)^{n+1}$$

Proof: Induction on n and the same method used for the second-order formula.

Example

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\cos(c)$ for some c between 0 and x.

The "method of exhaustion" for computing areas



If the upper (purple) area and the lower (pink) area approach a common limit when the region is partitioned into arbitrarily thin rectangles, then that limit is the (gray) area under the curve.

Assignment due next class

Practice exercises (not to hand in)

Compute the following limits

(a) by applying l'Hôpital's rule, and

(b) by applying Taylor's formula.

1. $\lim_{x \to 0} \frac{1 - \cos(x)}{x^2}$
2. $\lim_{x \to 0} \frac{e^x - \sin(x) - \cos(x)}{x^2}$
3. $\lim_{x \to \frac{\pi}{2}} \frac{1 - \sin(x)}{(x - \frac{\pi}{2})^2}$