Announcements

- I have posted the homework averages in eCampus.
- Our final class meeting is tomorrow (May 1).
- I will hold my usual office hour 3:00–4:00 in the afternoon on Tuesday (May 1) and Thursday (May 3).

About the final exam

- The comprehensive final examination takes place 8:00–10:00 on Monday morning (May 7).
- Material for the final exam: sections 0.3, 1.1–1.4, 2.1–2.5, 2.6.1, 2.6.2, 3.1–3.4, 4.1–4.4, 5.1–5.3.
- The exam has 7 problems (in the same style as the midterm exams).
- Please bring your own paper to the exam to work on.

Recap on integrable functions

For a partition $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$, and $M_k := \sup\{f(x) : x_{k-1} \leq x \leq x_k\}$, the corresponding *upper sum* of f on the interval $[a, b]$ is $\sum_{k=1}^{n} M_k(x_k - x_{k-1})$.

Similarly define lower sums by replacing sup with inf.

If the infimum of all the upper sums equals the supremum of all the lower sums, then f is an *integrable function* on the interval $[a, b]$.

Theorem (Cauchy)

If f is continuous on $[a, b]$, then f is integrable.

Properties of the integral

1. Linearity:

$$\int_a^b \lambda f(x) + \mu g(x)\, dx = \lambda \int_a^b f(x)\, dx + \mu \int_a^b g(x)\, dx.$$

2. Preservation of order:

$$\text{If } f(x) \leq g(x) \text{ for all } x\text{, then } \int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx.$$

3. Absolute value: $\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx.$

4. Interval decomposition:

$$\text{If } a < b < c\text{, then } \int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx.$$

Fundamental theorem of calculus, part I

Theorem

If f is continuous, and $F(x) = \int_a^x f(t)\,dt$, then F is differentiable, and $F'(x) = f(x)$.

Example.

What is the derivative of $\int_x^b f(t)\,dt$?

Solution.

Rewrite $\int_x^b f(t)\,dt$ as $\int_a^b f(t)\,dt - \int_a^x f(t)\,dt$

to see that the derivative is $-F'(x)$ or $-f(x)$. $\square$

Example of the fundamental theorem combined with the chain rule

If $G(x) = \int_{x^3}^{\sin(x)} \sqrt{1+t^2}\,dt$, find $G'(x)$.

Solution.

Combine the chain rule with the fundamental theorem of calculus.

Rewrite the problem as $\int_{x^3}^{c} + \int_{c}^{\sin(x)}$.

Then by the chain rule, the derivative is

$$-\sqrt{1+x^6} \cdot 3x^2 + \sqrt{1+\sin^2(x)} \cdot \cos(x).$$

□

Fundamental theorem of calculus, part II

Theorem

Suppose f is a continuous function, and F is a differentiable function such that $F'(x) = f(x)$ for all x; that is, F is an antiderivative of f. Then $\int_a^b f(t)\,dt = F(b) - F(a)$.