Examination 2

Instructions: Please write your solutions on your own paper. These problems should be treated as essay questions to answer in complete sentences.

1. For each part, give an example of a subset of $\mathbb{R}$ satisfying the specified property.

   a) An unbounded open set whose complement is unbounded too.

   Solution. One example is $\mathbb{R} \setminus \mathbb{Z}$, the complement of the set of integers. This set consists of a union of open intervals, hence is open; and the set contains arbitrarily large numbers, hence is unbounded. The complementary set is the set of integers, evidently unbounded too.

   b) A non-empty compact set having empty interior.

   Solution. The simplest example is a singleton set, such as $\{4\}$. This set is closed and bounded and contains no intervals, hence is compact with empty interior.

2. Suppose $f : (0, 1) \to \mathbb{R}$ is defined as follows:

$$f(x) = \sqrt{x}, \qquad 0 < x < 1.$$

(You know from Section 2.8 that every positive real number has a unique positive square root, so f is well defined.) Prove the unsurprising fact that

$$\lim_{x\to 0} f(x) = 0.$$

Solution. Fix a positive ε, and set δ equal to ε^2. If x lies in the interval $(0, 1)$, and $|x-0| < \delta$, then $|f(x) - 0| = \sqrt{x} < \sqrt{\delta} = \varepsilon$. Thus the ε–δ definition of limit is satisfied.

An alternative argument could be to say that on the larger domain $[0, 1]$, the function x^2 is certainly continuous, for the identity function x is continuous, and the product of two continuous functions is continuous. And x^2 is a strictly monotonic bijection of the interval $[0, 1]$ onto itself. The theorem about inverse functions on intervals (stated below in the extra-credit problem) implies that the inverse function $\sqrt{x}$ is continuous on the interval $[0, 1]$. Continuity of a function f on the closed interval implies that $\lim_{x\to 0} f(x) = f(0)$. Taking f to be the square-root function gives the required conclusion.

3. Give an example of a function $f : (0, 1) \to \mathbb{R}$ that is continuous at every point of the interval $(0, 1)$ but is not uniformly continuous on this interval. Explain why your example works.

Solution. Notice that if f extends to be a continuous function on the compact interval $[0, 1]$, then a theorem implies that f is uniformly continuous on the interval $[0, 1]$, hence also on the smaller interval $(0, 1)$. Therefore the required example must fail to extend continuously to one of the endpoints. Here are two examples.

1. Suppose $f(x) = 1/x$. Being the reciprocal of a continuous function that is never equal to zero, the function f is continuous at each point of the domain $(0, 1)$.

To see that f is not uniformly continuous on the interval, observe that

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|x-y|}{xy} \geq \frac{|x-y|}{x}$$

when x and y are points of the interval $(0, 1)$. Think of x as a fixed base point. If a positive target ε is specified, then the property that $|f(x) - f(y)| < \varepsilon$ can hold only if $|x - y| < \varepsilon x$. Therefore εx is the largest possible δ that can work in the definition of continuity at x. Since the value of εx becomes arbitrarily close to 0 when x approaches 0, there cannot be a single positive δ that works for every x simultaneously.

2. Another example is $\sin(\pi/x)$. Being the composition of two continuous functions, this function is continuous at each point of the interval $(0, 1)$. And the function even has bounded image.

Now this function takes the value 0 when $x = 1/n$ (where $n = 2, 3, \dots$) and takes the value 1 when $x = 2/(4n + 1)$. Thus there are values of x arbitrarily close together at which the values of the function differ by 1: the definition of uniform continuity is violated when $\varepsilon = 1/2$.

4. State **one** of the following theorems (your choice):

   a) the intermediate-value theorem, or

   b) the extreme-value theorem, or

   c) the Heine–Borel covering theorem.

Solution. The intermediate-value theorem is Theorem 6.1.2 on page 99. The extreme-value theorem is Corollary 6.3.2 on page 102. The Heine–Borel covering theorem is a combination of Definition 4.5.5 and Theorem 4.5.6 on page 77.

5. Suppose $f : (0, 1) \to \mathbb{R}$, and let S denote the set $\{\, x \in (0, 1) : f \text{ is continuous at } x \,\}$. Must S be an open set? Supply a proof or a counterexample, as appropriate.

Solution. If the function f has only a finite number of discontinuities, then the set S is the complement of a finite set, hence is open. To find a counterexample requires considering a function that has infinitely many points of discontinuity. Here are three examples.

Examination 2

1. Define $f(x)$ to be equal to zero unless x has the form $\frac{1}{2} + \frac{1}{10^n}$ for some positive integer n, and set $f\left(\frac{1}{2} + \frac{1}{10^n}\right)$ equal to $\frac{1}{10^n}$. Thus $f(0.51) = 0.01$, and $f(0.501) = 0.001$, and so on. Evidently f is discontinuous at each point $\frac{1}{2} + \frac{1}{10^n}$, since $f(x)$ takes the value 0 at values of x arbitrarily close to $\frac{1}{2} + \frac{1}{10^n}$. On the other hand, $\lim_{n\to\infty} \frac{1}{10^n} = 0$, so f is continuous at the point $\frac{1}{2}$. Accordingly, the set S fails to be open, since $\frac{1}{2} \in S$, but S contains no interval around $\frac{1}{2}$.

2. Suppose

$$f(x) = \begin{cases} x - \frac{1}{2}, & \text{if } x \text{ is a rational number,} \\ 0, & \text{if } x \text{ is an irrational number.} \end{cases}$$

Since every real number c between 0 and 1 is a limit of a sequence of irrational numbers, the second clause implies that if $\lim_{x\to c} f(x)$ exists, then the limit must be 0. On the other hand, the first clause implies that $\lim_{x\to c} f(x)$ can be 0 only if $c = 1/2$. The upshot is that the set S of points of continuity is the singleton set $\{1/2\}$, evidently not an open set.

3. This example is similar to one from class on March 21. Suppose

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is an irrational number,} \\ \frac{1}{n}, & \text{if } x = m/n, \text{ the integers } m \text{ and } n \text{ having no common factor.} \end{cases}$$

The irrational numbers are dense, so f takes the value 0 in every neighborhood of every rational number. But the value of f at a rational number is not 0, so f is discontinuous at every rational number. On the other hand, when a sequence of rational numbers converges to an irrational limit, the denominators of the rational numbers must blow up, so the value of f tends to 0. Therefore f is continuous at every irrational number. Thus S is the set of irrational numbers between 0 and 1, evidently not an open set, for S contains no intervals.

Remark. Although the set S defined in the problem is not necessarily an open set, the set S is always the intersection of a sequence of open sets.

6. Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are two continuous functions. Prove that if $f(r) = g(r)$ for every rational number r, then $f(x) = g(x)$ for every real number x.

Solution. The rational numbers are dense, so for an arbitrary real number x there exists a sequence (r_n) of rational numbers such that $\lim_{n\to\infty} r_n = x$. Continuous functions preserve convergence of sequences, so

$$f(x) = \lim_{n\to\infty} f(r_n) \overset{\text{by hypothesis}}{=} \lim_{n\to\infty} g(r_n) = g(x).$$

Thus $f(x) = g(x)$ for an arbitrary real number x, as required.

Examination 2

Extra Credit Problem. A theorem about inverse functions says that if I and J are intervals in $\mathbb{R}$, and a function f is a continuous bijection from I onto J, then the inverse function f^{-1} is automatically continuous on J.

Your task is to construct an example of two subsets A and B of $\mathbb{R}$ and a bijective continuous function f from A onto B such that f^{-1} is discontinuous at some point of B. (In view of the theorem, your sets A and B cannot both be intervals.)

Solution. Here are two examples.

1. Let A be the union of the intervals $[0, 1)$ and $[2, 3)$, let B be the interval $[0, 2)$, and define a function f as follows:

$$f(x) = \begin{cases} x, & \text{if } 0 \le x < 1, \\ x - 1, & \text{if } 2 \le x < 3. \end{cases}$$

Evidently f is a continuous, strictly increasing function that maps the disconnected domain A onto the image B. The inverse function f^{-1} is discontinuous at the point 1 in B, for points slightly to the left of 1 in B map to points close to 1 in A, while points slightly to the right of 1 in B map to points close to 2 in A. Even without putting your finger on the point 1, you could infer from the intermediate-value theorem that f^{-1} must have a point of discontinuity.

2. Let A be $\mathbb{Z}$, the set of integers. Since this set has no limit point, *every* function with domain A is continuous by default. Define an injective function f as follows:

$$f(n) = \begin{cases} 0, & \text{if } n = 0, \\ \frac{1}{n}, & \text{if } n \ne 0. \end{cases}$$

Let B be the image set, $\{0\} \cup \left\{ \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \dots \right\}$. Then $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ is a sequence in B that converges to the point 0 in B, and the inverse function f^{-1} maps this convergent sequence to the divergent sequence $(n)_{n=1}^{\infty}$, so f^{-1} is not continuous at 0.