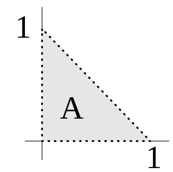
Math 436 **Exam 2** **8 November 1993**

1. (30 points) Consider the topological space $(X, \mathcal{T})$, where $X = \{a, b, c\}$, and $\mathcal{T} = \{\varnothing, X, \{a\}, \{b, c\}\}$. There are $2^3 = 8$ subsets of X: namely, $\varnothing$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{b, c\}$, $\{a, c\}$, and $\{a, b, c\}$. Which of these eight subsets of X are connected, and which are disconnected? Explain why. (Hint: remember that a subset A is connected if and only if the topological space $(A, \mathcal{T}_A)$ is connected.)

2. (30 points) In the product space $\mathbb{R} \times \mathbb{R}$, consider the subset A defined by $A = \{\,(x, y) : x > 0,\ y > 0, \text{ and } x + y < 1\,\}$, as shown in the diagram. Determine the closure of A in $\mathbb{R} \times \mathbb{R}$ (with the product topology), and justify your answer, when



   (i) both copies of $\mathbb{R}$ carry the half-open interval $\mathcal{H}$ topology;

   (ii) both copies of $\mathbb{R}$ carry the open half-line $\mathcal{C}$ topology;

   (iii) the first copy of $\mathbb{R}$ carries the discrete $\mathcal{D}$ topology, and the second copy of $\mathbb{R}$ carries the trivial indiscrete $\mathcal{I}$ topology.

   (Hint: remember that a point p belongs to the closure of A if and only if every neighborhood of p intersects A.)

3. (15 points)

   (i) Define what it means for a topological space to be connected.

   (ii) State another property that is equivalent to connectedness.

   (iii) State yet another property that is equivalent to connectedness.

In the next four questions, give a brief explanation if the answer is "Yes", and exhibit a counterexample if the answer is "No". (6 points each)

4. If X and Y are topological spaces, $f : X \to Y$ is a homeomorphism, and B is a connected subset of Y, must $f^{-1}(B)$ be a connected subset of X?

5. Is the additive inverse function $i : \mathbb{R} \to \mathbb{R}$ defined by the formula $i(x) = -x$ an $\mathcal{H}$-$\mathcal{C}$ continuous function?

6. If X and Y are topological spaces, does the collection of subsets of $X \times Y$ of the form $U \times V$, where U is an open subset of X and V is an open subset of Y, form a subbase for the product topology on $X \times Y$?

7. Is it true that a topological space X is disconnected if and only if every subset of X is both open and closed?

Extra credit (6 points):

8. Consider the infinite product space $\mathbb{R}^\omega$, which may be viewed as the space of all sequences $(x_1, x_2, \dots)$ of real numbers. Let A be the subset of $\mathbb{R}^\omega$ consisting of all convergent sequences, that is, sequences such that $\lim_{j \to \infty} x_j$ exists. Determine the interior of A in $\mathbb{R}^\omega$ and the closure of A in $\mathbb{R}^\omega$ (where $\mathbb{R}^\omega$ carries the product topology). Explain your answers.