New concepts in section 3.2

- neighborhood [Definition 3.2.1]
- interior of a set [Exercise 3.2.5]
- separable space [Exercise 3.2.4]
- second axiom of countability [Exercise 2.2.4]
- the Sorgenfrey line [Exercise 3.2.11)

Neighborhood

A set S is a *neighborhood* of a point x if S contains an open set that contains x.

Example

In $\mathbb{R}$ with the Euclidean topology, the interval $[0, 2]$ is a neighborhood of the point 1 but is not a neighborhood of the point 0.

A set is open if and only if the set contains a neighborhood of each of its points.

Interior

The *interior* of a set S is the union of all open subsets of S.

Example

In $\mathbb{R}$ with the Euclidean topology, the interior of the interval $[0, 2)$ is the interval $(0, 2)$.
The interior of $\mathbb{Q}$ is $\varnothing$.

Separable space

A topological space is *separable* if it contains a countable dense subset.

Example

The space $\mathbb{R}$ with the Euclidean topology is separable because $\mathbb{Q}$ is a countable dense subset.

The space $\mathbb{R}$ with the discrete topology is not separable. The only dense subset is $\mathbb{R}$ itself, which is uncountable.

Second axiom of countability

A topological space is *second countable* when there exists a countable basis for the topology.

Example

- $\mathbb{R}$ with the Euclidean topology is second countable: The open intervals having rational endpoints form a basis.
- $\mathbb{R}$ with the discrete topology is not second countable: Every basis must contain every singleton set.

Sorgenfrey line [named for Robert Sorgenfrey (1915–1995)]

The "half-open" intervals of the form $[a, b)$ form a basis for a certain topology on $\mathbb{R}$. This topological space is known as the Sorgenfrey line.

The rational numbers are a dense subset of this topological space, so the Sorgenfrey line is separable.

The Sorgenfrey line is not second countable.

Assignment due next class

1. Suppose $\mathcal{B} = \{\, [a, b) : a \in \mathbb{R}, b \in \mathbb{Q}, \text{ and } a < b \,\}$,
   $\mathcal{B}_1 = \{\, [a, b) : a \in \mathbb{R}, b \in \mathbb{R}, \text{ and } a < b \,\}$, and
   $\mathcal{B}_2 = \{\, [a, b) : a \in \mathbb{Q}, b \in \mathbb{Q}, \text{ and } a < b \,\}$. Show that $\mathcal{B}$ and $\mathcal{B}_1$ are two different bases for the same topology on $\mathbb{R}$. Is $\mathcal{B}_2$ another basis for the same topology?
2. Write a solution to number 2 in Exercises 3.1.
3. Read section 3.3 in the textbook (about connectedness).