Recap: compactness

A topological space X is compact when for every collection of open subsets whose union equals X, there is a finite subcollection whose union equals X.

A subset S of a topological space is compact when S is compact in the subspace topology, in other words, every open cover of S admits a finite subcover.

Compactness rephrased in terms of closed sets

A collection of sets has the *finite-intersection property* if every finite subcollection of these sets has nonempty intersection.

Example: the collection of intervals $(0, x)$ as x runs over the positive real numbers.

Theorem

A topological space is compact if and only if every collection of **closed** *sets having the finite-intersection property also has nonempty intersection.*

Compactness is preserved by continuous mappings

If A is a compact subset of a topological space X_1, and $f\colon X_1 \to X_2$ is a continuous mapping, then $f(A)$ is a compact subset of X_2.

Proof.

If $\{U_j\}_{j\in J}$ is an open cover of $f(A)$ in X_2, then $\{f^{-1}(U_j)\}_{j\in J}$ is an open cover of A in X_1.
Compactness of A yields a finite set K such that $\{f^{-1}(U_j)\}_{j\in K}$ is a finite open cover of A.
Then $\{U_j\}_{j\in K}$ is a finite open cover of $f(A)$. $\square$

Assignment due next class

Write solutions to numbers 4 and 6 in Exercises 7.1.