Announcements/reminders

- I will hold my usual office hour 3:00–4:00 in the afternoon on Tuesday (May 1) and Thursday (May 3).
- The comprehensive final examination takes place 10:30–12:30 on Friday (May 4).
- Material for the final exam: Chapters 1–5, Sections 6.1–6.2, and Chapter 7.
- The six exam questions are mostly definitions, examples, and theorems.
- Please bring paper to the exam.

An exercise on quotient spaces

Let X be $\mathbb{R}$ with the standard Euclidean topology.

Form a quotient space Y by identifying all the integers.

The space Y can be viewed as $(\mathbb{R} \setminus \mathbb{Z}) \cup \{\mathfrak{z}\}$, where $\mathfrak{z}$ represents the equivalence class of the integers.

Is this quotient space Y

1. connected?
2. compact?
3. Hausdorff?
4. separable?
5. path-connected?
6. second countable?

Answer to the exercise

For 1–5, same as $\mathbb{R}$: yes for 1, 3, 4, 5 and no for 2.

But 6 is true in $\mathbb{R}$ yet false in Y. There is no countable base [defined in Exercise 6.1.11] for the special point $\mathfrak{z}$ in Y.

Proof.

Suppose $\{U_n\}_{n=-\infty}^{\infty}$ is a countable family of open sets in Y containing $\mathfrak{z}$ and indexed by the integers.
For each integer n, there is some number ε_n $(0 < \varepsilon_n < 1/2)$ such that $(n - \varepsilon_n, n + \varepsilon_n) \subset U_n$. Define $V = \bigcup_{n=-\infty}^{\infty}(n - \frac{\varepsilon_n}{2}, n + \frac{\varepsilon_n}{2})$.
Then V is a neighborhood of $\mathfrak{z}$ that contains no U_n. □