Examination 1

1. Define
   (a) the concept of the closure of a set, and
   (b) the concept of a basis for a given topology.

   Solution. The closure of a set A can be defined as the smallest closed set containing A, or as the intersection of all closed sets containing A, or as the union of A with the set of limit points of A.

   A basis for a given topology τ is a subset σ of τ such that every element of τ can be obtained as a union of elements of σ. In alternative language, a basis for a given topology is a collection of open sets such that every open set can be obtained by taking a union of some open sets belonging to the basis.

2. Give an example of a separable Hausdorff topological space (X, τ) with the property that X has no limit points.

   Solution. A point x is a limit point of X when every neighborhood of x intersects $X \setminus \{x\}$. This property evidently holds unless the singleton set $\{x\}$ is an open set. Saying that X has no limit points is equivalent to saying that every singleton set is an open set. In other words, the topology on X is the discrete topology.

   The discrete topology is automatically Hausdorff, for if x and y are distinct points, then the sets $\{x\}$ and $\{y\}$ are disjoint open sets that separate x from y. A minimal basis for the discrete topology consists of all singletons, so the discrete topology is separable if and only if the space X is countable.

   Thus every countable set with the discrete topology is an example. Moreover, all examples have this form. For a concrete example, take the set of natural numbers with the discrete topology. Another example is the set of Aggies (a finite set) with the discrete topology. For an extreme example, take X to be the singleton space {Texas} with the topology $\{\varnothing, X\}$.

3. Suppose (X, τ) is a topological space, and Y is a dense open subset equipped with the subspace topology. If the space X is connected, must Y be connected too? Provide a proof or a counterexample, whichever is appropriate.

   Solution. For a counterexample, take X to be $\mathbb{R}$ (the set of real numbers) with the standard Euclidean topology. This space is known to be connected [Proposition 3.3.5]. Take Y to be $\mathbb{R} \setminus \{0\}$. Being the union of two disjoint unbounded open intervals, the set Y is a disconnected open subset of $\mathbb{R}$, and Y is dense because the excluded point 0 is a limit point of Y (so the closure of Y is all of $\mathbb{R}$).

   For an alternative counterexample, based on an ad hoc construction, take X to be the set $\{a, b, c\}$ with topology τ equal to $\{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$. This space is connected: none of the proper open subsets $\{a\}$, $\{b\}$, and $\{a, b\}$ is the complement of another open set, so the only clopen subsets are $\varnothing$ and X. Let Y be the open subset $\{a, b\}$. This set is dense in X because it intersects every nonempty open subset of X [Proposition 3.1.15]. The subspace topology on Y is $\{\varnothing, Y, \{a\}, \{b\}\}$. The subspace Y is disconnected because $Y = \{a\} \cup \{b\}$, and the sets $\{a\}$ and $\{b\}$ are disjoint open subsets of Y.

4. The textbook defines a topological space to be a T_2 space if every two distinct points admit disjoint open neighborhoods. Prove that this property is equivalent to the following: for every two distinct points x and y, there exists an open set V such that $x \in V$ and $y \notin \overline{V}$ (where $\overline{V}$ denotes the closure of V).

   Solution. If the indicated property holds, then the complement of the closed set $\overline{V}$ is an open set containing y, and this set is disjoint from the open set V that contains x. So distinct points of X admit disjoint open neighborhoods.

   For the converse, suppose that X is Hausdorff, and let x and y be distinct points of X. There exist disjoint open sets V and U such that $x \in V$ and $y \in U$. Since U is a neighborhood of y that does not intersect V, the point y is not a limit point of V. Also $y \notin V$, for $V \cap U = \varnothing$. Since the closure of V is the union of V with the set of limit points of V, the preceding observations show that $y \notin \overline{V}$.

5. Suppose (X_1, τ_1) is the set of rational numbers equipped with the usual topology induced by the Euclidean topology on $\mathbb{R}$, and (X_2, τ_2) is the set of natural numbers equipped with the usual topology induced by the Euclidean topology on $\mathbb{R}$. Are the topological spaces (X_1, τ_1) and (X_2, τ_2) homeomorphic to each other? Explain why or why not.

   Solution. For every natural number n, the singleton set $\{n\}$ is an open subset of $\mathbb{N}$ in the subspace topology induced by the Euclidean topology on $\mathbb{R}$, since $\{n\}$ is the intersection of $\mathbb{N}$ with an open interval in $\mathbb{R}$: namely, $\{n\} = \mathbb{N} \cap (n-1, n+1)$. On the other hand, singleton subsets of $\mathbb{Q}$ are not open in the subspace topology on $\mathbb{Q}$: the intersection of $\mathbb{Q}$ with a nonempty open interval in $\mathbb{R}$ has infinitely many points.

   Since homeomorphisms map open sets to open sets, the two spaces (X_1, τ_1) and (X_2, τ_2) are not homeomorphic: singleton sets are open in one space but not in the other space.

Optional Extra Credit Problem

Notation. Let (X, τ) be a topological space. When A is a subset of X, let A^- denote the closure of A, and let A^c denote the complement of A (that is, $A^c = X \setminus A$).

Examination 1

Problem. Let A be a subset of X. Prove that A is equal to the closure of some open set if and only if $A = A^{c-c-}$ (the closure of the complement of the closure of the complement).

Solution. The set A^{c-} is closed, so the complementary set A^{c-c} is open, so the set A^{c-c-} is the closure of an open set. So if $A = A^{c-c-}$, then A is indeed the closure of an open set.

To establish the other direction, suppose that A is the closure of some open set B. Represent the set $A \setminus B$ as the union of two disjoint sets: namely, let C be the set of points of $A \setminus B$ that are *not* limit points of A^c, and let D be the set of points of $A \setminus B$ that *are* limit points of A^c. The four sets B, C, D, and A^c form a partition of the whole space X into pairwise disjoint subsets.

The set A^{c-} is the union of A^c and the set of limit points of A^c. By construction, no point of C is a limit point of A^c. And no point of B is a limit point of A^c, for B is a neighborhood of each of its points and does not intersect A^c. On the other hand, all points of D are limit points of A^c by construction. Thus $A^{c-} = A^c \cup D$.

The complementary set A^{c-c} is therefore equal to $B \cup C$. It remains to determine the closure of this set. Now

$$B \subseteq B \cup C \subseteq A, \qquad \text{so}$$
$$B^- \subseteq (B \cup C)^- \subseteq A^-.$$

But $B^- = A$, and $A^- = A$ because A is closed. Therefore $A \subseteq (B \cup C)^- \subseteq A$, so $(B \cup C)^- = A$. In other words, $A^{c-c-} = A$, as required.

Historical note. The source for this statement is the doctoral thesis of the renowned Polish mathematician Kazimierz (Casimir) Kuratowski (1896–1980). Published as "Sur l'opération $\overline{A}$ de l'Analysis Situs" [*Fundamenta Mathematicae* **3** (1922) 182–199], the first part of Kuratowski's dissertation develops topology axiomatically based on the closure operation. Kuratowski deduced from the indicated statement that by repeatedly applying the operations of forming complements and forming closures, it is possible to create from a given set A at most 14 distinct sets (including the original set A).

Remark. The 14 sets that can be constructed from A by using closure and complement are:
A,
A^c and A^-,
A^{c-} and A^{-c},
A^{c-c} and A^{-c-},
A^{c-c-} and A^{-c-c},
A^{c-c-c} and A^{-c-c-},
A^{c-c-c-} and A^{-c-c-c},
$A^{c-c-c-c}$.
The next two sets in the list would be $A^{-c-c-c-}$ and $A^{c-c-c-c-}$, but because A^{-c-} and A^{c-c-} are closures of open sets, the proposition in the problem implies that $A^{-c-c-c-} = A^{-c-}$ and $A^{c-c-c-c-} = A^{c-c-}$, so no new sets actually arise.