Examination 2

1. Define
   (a) the meaning of a *component* of a topological space (X, τ),
   (b) the meaning of two metrics on a set X being *equivalent* metrics.

   Solution. A component of X is a maximal connected subset of X. Alternatively, a nonvoid subset C is a component of X if for some point x in C (equivalently, for every point x in C), the set C is the union of all connected subsets of X containing x.

   Two metrics on X are equivalent if they induce the same topology on X. Equivalently, metrics d_1 and d_2 are equivalent if every d_1 ball can be expressed as a union of d_2 balls, and every every d_2 ball can be expressed as a union of d_1 balls.

2. Give an example of a disconnected metric space (X_1, d_1) and a connected metric space (X_2, d_2) and a continuous mapping $f : X_1 \to X_2$.

   Solution. Here are three different examples.

   - $X_1 = X_2 = \mathbb{R}$, and d_1 is the discrete metric, and d_2 is the Euclidean metric, and f is the identity function: $f(x) = x$. Then (X_1, d_1) is disconnected, since every subset is clopen in the discrete topology; and (X_2, d_2) is connected (Proposition 3.3.5); and f is continuous because every function whose domain has the discrete topology is continuous.
   - $X_1 = \mathbb{R} \setminus \{0\}$, $X_2 = \mathbb{R}$; $d_1 = d_2 =$ Euclidean metric; and $f(x) = x^2$. Then X_1 is disconnected, being the union of two disjoint open pieces $(-\infty, 0)$ and $(0, \infty)$; and X_2 is connected (Proposition 3.3.5); and f is continuous because polynomials are continuous with respect to the Euclidean topology.
   - X_1 is the two-point space $\{a, b\}$ and X_2 is the singleton space $\{c\}$; the metrics d_1 and d_2 are the discrete metric; and f is the only available function: $f(a) = c = f(b)$. Then X_1 is disconnected, being the union of the two disjoint open subsets $\{a\}$ and $\{b\}$; and X_2 is connected because there is only one nonvoid open set; and f is continuous because the domain has the discrete topology (alternatively, because the codomain has the indiscrete topology).

   Remark. Proposition 5.2.1 says that the continuous image of a connected space is always connected. The preceding examples illustrate that disconnectedness is not preserved by continuous mappings.

3. Suppose X is the two-point space $\{a, b\}$ equipped with the topology consisting of the three sets $\varnothing$, X, and $\{a\}$. Is this topological space path-connected? Explain why or why not.

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Solution. The question amounts to asking whether there exists a path from a to b, in other words, a continuous function $f : [0,1] \to X$ such that $f(0) = a$ and $f(1) = b$. There is indeed such a function, for instance

$$f(x) = \begin{cases} a, & \text{when } 0 \le x < 1/2, \\ b, & \text{when } 1/2 \le x \le 1. \end{cases}$$

To verify that this function is continuous, observe that $f^{-1}(\varnothing) = \varnothing$ and $f^{-1}(X) = [0,1]$, and these two inverse images are automatically open subsets of $[0,1]$. The only other open subset of X is $\{a\}$, and $f^{-1}(\{a\}) = [0, 1/2)$. Now the set $[0, 1/2)$ is open in the standard topology that $[0,1]$ inherits as a subspace of $\mathbb{R}$, for $[0,1/2) = [0,1] \cap (-1, 1/2)$, and the interval $(-1, 1/2)$ is open in $\mathbb{R}$. In summary, the function f is continuous, so the space X is path connected.

4. Suppose $f(x) = \cos(x)$ when $x \in \mathbb{R}$. Show that if f is viewed as a mapping from the topological space ($\mathbb{R}$, finite-closed topology) into itself, then f is not continuous.

   Solution. In the finite-closed topology, the closed sets are the finite sets, along with the whole space. In particular, the singleton set $\{1\}$ is closed. Periodicity of the cosine function implies that $f^{-1}(\{1\})$ is the set $\{\, 2\pi n : n \in \mathbb{Z} \,\}$, which is not closed (being an infinite set). Therefore the function f is not continuous, for there exists a closed set whose inverse image is not closed.

5. Suppose X is the set of letters of the Greek alphabet. Prove that if d is an arbitrary metric on X, then the topology induced by d must be the discrete topology.

   Solution. What is significant about the Greek alphabet in this problem is merely that the alphabet is a finite set. Here are two arguments showing that an arbitrary metric d on a finite set X induces the discrete topology.

   Method 1. What needs to be shown is that if x is an arbitrary point of X, then the singleton set $\{x\}$ is open. If X has only one element, then there is nothing to show. If X has more than one element, but finitely many, then the collection of distances $\{\, d(x,y) : y \in X \setminus \{x\} \,\}$ is a finite nonvoid set of positive real numbers and so has a minimal element r. The open ball $B_r(x)$, or $\{\, y \in X : d(x,y) < r \,\}$, is identical to $\{x\}$. Open balls are open sets in the metric topology, so $\{x\}$ is indeed open, as required.

   Method 2. We know that every metric space is a Hausdorff (T_2) space (Proposition 6.1.25). And from Chapter 4, we know that every T_2 space is a T_1 space (Exercise 4.1.13 part iii). So in metric spaces, points (singleton sets) are closed. Finite unions of closed sets are closed, so finite subsets of metric spaces are closed. Consequently, if a metric space has only finitely many points, then every subset is closed. Taking complements shows that every subset is open too. Thus the topology is the discrete topology.

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Optional Extra Credit Problem

Given a collection of functions from a set X_1 to a set X_2, and given a topology τ_2 on X_2, there is a natural way to create a topology τ_1 on X_1: namely, the coarsest topology for which all the specified functions are continuous. (A topology τ is coarser than a topology σ if $\tau \subseteq \sigma$, that is, every τ-open set is also σ-open.)

Specifically, suppose that both X_1 and X_2 are $\mathbb{R}$, and τ_2 is the Euclidean topology. Consider the collection of step functions $\{ f_c : c \in \mathbb{R} \}$ of the following form:

$$f_c(x) = \begin{cases} 0, & \text{if } x < c, \\ 1, & \text{if } x \geq c. \end{cases}$$

Determine the coarsest topology τ_1 on X_1 that makes all of these step functions continuous as functions from (X_1, τ_1) to (X_2, τ_2).

Solution. The inverse image under f_c of the open interval $(-1, 1/2)$ is the unbounded interval $(-\infty, c)$, and similarly the inverse image under f_c of the open interval $(1/2, 2)$ is the unbounded interval $[c, \infty)$. Accordingly, both $(-\infty, c)$ and $[c, \infty)$ must belong to the topology τ_1 for every real number c.

The intersection of two sets belonging to a topology must belong to the topology. In particular, the intersection $[c_1, \infty) \cap (-\infty, c_2)$ must belong to τ_1 for all real numbers c_1 and c_2. When $c_1 < c_2$, this intersection is the interval $[c_1, c_2)$. Such intervals form a basis for the Sorgenfrey topology on $\mathbb{R}$. Thus the topology τ_1 must contain the Sorgenfrey topology.

On the other hand, the indicated step functions all are continuous when the domain has the Sorgenfrey topology. Indeed, if U is an open set in the codomain that contains neither of the values 0 and 1, then $f_c^{-1}(U) = \varnothing$, a set that is open in every topology. If U contains both of the values 0 and 1, then $f_c^{-1}(U)$ equals the whole domain, which is open in every topology. If U contains the point 0 but not the point 1, then $f_c^{-1}(U) = (-\infty, c) = \bigcup_{n=1}^{\infty} [c - n, c)$, a set that is open in the Sorgenfrey topology. And if U contains the point 1 but not the point 0, then $f_c^{-1}(U) = [c, \infty) = \bigcup_{n=1}^{\infty} [c, c + n)$, a set that is open in the Sorgenfrey topology.

The preceding paragraph implies that τ_1 is contained in the Sorgenfrey topology, and the second paragraph implies that τ_1 contains the Sorgenfrey topology. Accordingly, the Sorgenfrey topology is the coarsest topology that makes all the given step functions continuous.