Examination 2

1. Equip $\mathbb{R}$, the set of real numbers, with the standard topology (corresponding to the absolute-value metric). Let the subspace Y be the half-open interval $[0, 2)$, that is, $\{ x \in \mathbb{R} : 0 \leq x < 2 \}$. Let A be the interval $[0, 1)$. With respect to the subspace topology on Y, is the set A open, closed, both, or neither? Explain.

   Solution. Since the interval $(-1, 1)$ is open in $\mathbb{R}$, and $A = (-1, 1) \cap Y$, the set A is open with respect to the subspace topology.

   The closure of A in $\mathbb{R}$ equals $[0, 1]$, and the closure of A with respect to the subspace topology is equal to the intersection of Y with the closure of A in $\mathbb{R}$ (see Proposition 4 on page 68 in Section 4.2). Thus the set A is not closed with respect to the subspace topology, for the closure is the strictly bigger set $[0, 1]$.

2. Equip $\mathbb{N}$, the set of natural numbers, with the cofinite topology (that is, the proper closed sets are the finite sets). With respect to the corresponding product topology on the product space $\mathbb{N} \times \mathbb{N}$, is the "diagonal" subset

   $$\{ (n, n) \in \mathbb{N} \times \mathbb{N} : n \in \mathbb{N} \}$$

   open, closed, both, or neither? Explain.

   Solution. The diagonal is neither open nor closed with respect to the product topology. More is true: the diagonal has no interior points, hence in a strong way fails to be an open set; and the closure of the diagonal is the whole space, so the diagonal in a strong way fails to be a closed set.

   Indeed, a basic open set in the product topology has the form $U \times V$, where U and V are open sets in the cofinite topology. If both U and V are nonempty sets, and k denotes the maximum of the finitely many natural numbers that are missing from either U or V, then the set $U \times V$ contains all pairs (m, n) for which both numbers m and n exceed k. In particular, since m and n can be different from each other (say $m = k + 1$ and $n = k + 2$), the set $U \times V$ contains some points that are not on the diagonal. Thus the diagonal contains no basic open set: the diagonal has empty interior.

   Taking complements shows that for every proper closed set in the product space, there exists a natural number k such that the given closed set contains no pair (m, n) for which both numbers m and n exceed k. In particular, a proper closed set contains only finitely many points of the diagonal.

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Therefore the only closed set containing the diagonal is the whole space: the diagonal is a dense subset of the product space!

Remark This product topology is *not* the cofinite topology. Finite sets are indeed closed, but there are some infinite closed sets too. For example, the infinite set $\{1\} \times \mathbb{N}$ is a closed set with respect to the product topology. Thus the product topology is strictly finer than the cofinite topology on the product space.

On the other hand, if the diagonal is considered as a subspace of the product space, then the subspace topology on the diagonal is the cofinite topology. Indeed, Exercise 7 on page 89 in Section 4.6 says that the diagonal is homeomorphic to $\mathbb{N}$.

3. Let X be $\{ x \in \mathbb{R} : 0 < x \}$ (the set of positive real numbers) equipped with the discrete topology, and let $f : X \to X$ be defined by setting $f(x)$ equal to x^2 for each value of x. Is this function f a homeomorphism? Explain why or why not.

   Solution. Yes, the function f is a homeomorphism (a continuous bijection with a continuous inverse). On the positive real numbers, the squaring function has an inverse (the square-root function), so f is a bijection (one-to-one and onto). Both f and its inverse are continuous because *every* function is continuous with respect to the discrete topology.

4. Prove that if a topological space satisfies the separation property T_4 and also satisfies the separation property T_1, then the space necessarily satisfies the separation property T_2.

   Solution. In a T_1 space, points (that is, singleton sets) are closed. In a T_4 space, two arbitrary disjoint closed sets can be included in disjoint open sets. Consequently, if a space has both properties T_1 and T_4, two arbitrary distinct points can be included in disjoint open sets. The latter property is precisely property T_2.

   Second solution, using continuous functions Let x and y be two distinct points of a topological space X that satisfies properties T_1 and T_4. By property T_1, the singleton sets $\{x\}$ and $\{y\}$ are disjoint closed sets. By property T_4 (in the guise of Urysohn's lemma), there exists a continuous function $f : X \to \mathbb{R}$ such that $f(x) = 0$ and $f(y) = 1$. Then $f^{-1}((-1/4, 1/4))$ is an open subset of X containing x, and $f^{-1}((3/4, 5/4))$ is an open subset of X containing y, and these two open sets are disjoint. Since x and y are arbitrary distinct points of X, property T_2 holds for the space X.

5. State either Urysohn's Lemma or Tietze's Extension Theorem.

   Solution. See Propositions 10 and 11 in Section 5.5 (pages 107 and 110).

6. Does there exist a topology τ on the set $\mathbb{R}$ of real numbers that makes $(\mathbb{R}, \tau)$ into a *compact* topological space? Explain why or why not.

   Solution. For an arbitrary topological space X, the trivial topology $\{X, \varnothing\}$ makes the space compact, because every open cover has a subcover consisting of a single set (namely, the whole set X). In particular, the trivial topology makes $\mathbb{R}$ into a compact space.

   Another example of a topology that makes an arbitrary space compact is the cofinite topology.

7. With respect to the standard topology on the real numbers, there does *not* exist a function $f : [0, 1] \to (0, 1)$ that is simultaneously continuous and surjective (onto). Why not?

   Solution. The image of a compact set under a continuous function is necessarily compact (Proposition 12 on page 157 in Section 7.4). The interval $[0, 1]$ is compact, but the interval $(0, 1)$ is not compact (with respect to the standard topology on the real numbers). Therefore the interval $(0, 1)$ cannot be the image of the interval $[0, 1]$ under a continuous function. If f is continuous, then the image of the compact interval $[0, 1]$ under f must be a proper subset of the noncompact interval $(0, 1)$.

8. Give an example of a topological space that is first countable but not second countable. Explain why your example works.

   Solution. One example is the set of real numbers with the discrete topology (Example 6 on page 145 in Section 7.2).

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If x is an arbitrary real number, then the singleton set $\{x\}$ is open with respect to the discrete topology, and every neighborhood of x includes the set $\{x\}$. Therefore $\{\{x\}\}$ is a neighborhood system of x that is countable (indeed, finite). Since x is arbitrary, the space is first countable.

On the other hand, suppose $\mathcal{B}$ is a basis for the topology. If x is an arbitrary real number, then the open set $\{x\}$ is a union of sets in the basis $\mathcal{B}$. But $\{x\}$ has no proper nonempty subset, so $\{x\}$ must be one of the sets in $\mathcal{B}$. Since x is arbitrary, and there are uncountably many real numbers, the basis $\mathcal{B}$ is uncountable. Thus the space is not second countable.

A more subtle example is the set of real numbers with the half-open interval topology (Example 7 on page 150 in Section 7.2). This space is first countable because $\{\, [x, x + 1/n) : n \in \mathbb{N} \,\}$ is a countable neighborhood basis of an arbitrary point x. On the other hand, if $\mathcal{B}$ is a basis for the topology, and x is an arbitrary real number, then the open set $[x, x + 1)$ is a union of basis elements, and one of these basis elements must contain x but no real number less than x. Thus there is an injective mapping from $\mathbb{R}$ into $\mathcal{B}$, so $\mathcal{B}$ is uncountable. Since every basis is uncountable, the topological space is not second countable.

Bonus Problem for Extra Credit:

Prove the following version of Cantor's nested-set theorem: If X is a Hausdorff topological space, and $\{K_j\}_{j=1}^{\infty}$ is a decreasing sequence of nonempty compact subsets of X (that is, $K_1 \supset K_2 \supset \cdots$), then the intersection $\bigcap_{j=1}^{\infty} K_j$ is not empty.

Solution. In a Hausdorff space, compact sets are closed (this statement is Proposition 11 on page 156 in Section 7.4), so each set K_j is closed in X. Since K_1 is a closed set, each set K_j is closed in K_1 with respect to the subspace topology on K_1 (by Proposition 4 on page 68 in Section 4.2, for example).

View K_1 as a topological space with the subspace topology inherited from X. Then K_1 is a compact space, since K_1 is a compact subset of X. By Proposition 8 on page 153 in Section 7.3, the compact space K_1 has the finite intersection property: if a family of closed subsets of K_1 has the property that every finite subfamily has nonempty intersection, then the whole family has nonempty intersection. Since the sets K_j are nested and nonempty, the intersection of a finite collection of these sets is nonempty (being equal to the set in the finite collection with the largest subscript). The finite intersection property implies that the intersection $\bigcap_{j=1}^{\infty} K_j$ is not empty.