

## Subharmonic functions and Perron's method

The goal of this exercise is to augment your understanding of subharmonic functions and their role in the solution of the Dirichlet problem by Perron's method.

### Entire subharmonic functions

1. Give an example of a continuous function that is subharmonic on the entire plane  $\mathbb{C}$ , harmonic on no open subset of  $\mathbb{C}$ , and (a) not bounded below; (b) non-negative, radial, and not convex.
2. Prove the following version of Liouville's theorem for subharmonic functions: if a function that is subharmonic on the entire plane  $\mathbb{C}$  is bounded above, then the function must be constant.  
Hint: make use of the comparison functions  $A + B \log |z|$ , which are harmonic on every annulus centered at the origin for every choice of the constants  $A$  and  $B$ .

### Perron's family on the punctured disc

As indicated in the textbook, the Dirichlet problem is not solvable on the punctured unit disc. The lack of a barrier function (a subharmonic peaking function) at the origin means that one loses control of the Perron family there.

The Perron family for the punctured disc  $\{z \in \mathbb{C} : 0 < |z| < 1\}$  with given boundary values is the family of all subharmonic functions on the punctured disc whose limsup at the boundary is less than or equal to the prescribed boundary values.

3. (a) If the boundary values are 0 at  $z = 0$  and 1 when  $|z| = 1$ , what is the pointwise supremum of the corresponding Perron family on the punctured disc?  
(b) If the boundary values are 1 at  $z = 0$  and 0 when  $|z| = 1$ , what is the pointwise supremum of the corresponding Perron family on the punctured disc?