Instructions. This take-home examination is due at the beginning of class on Monday, October 27. (There will be no class meeting on Friday, October 24.)

In working on the examination, you may not consult other sentient beings. You may, however, consult your textbook.

1. Consider the rational function f defined by $f(z) = \dfrac{6}{z(z-1)(z-2)}$.
   (a) Find a series expansion for f in powers of z and $1/z$ that converges in the punctured disc $\{ z : 0 < |z| < 1 \}$.
   (b) Find a series expansion for f in powers of z and $1/z$ that converges in the annulus $\{ z : 1 < |z| < 2 \}$.
   (c) Find a series expansion for f in powers of z and $1/z$ that converges in the unbounded region $\{ z : 2 < |z| \}$.

2. Use contour integration to prove that if n is an integer bigger than 2, then
$$\int_0^\infty \frac{x\,dx}{x^n+1} = \frac{\pi/n}{\sin(2\pi/n)}.$$
A complete solution will include both a residue computation and a limiting argument. [Hint: this problem is easier to solve while eating a slice of pizza.]

3. The goal of this problem is to demonstrate the existence of the "partial fractions" decomposition of a rational function in a special case. Suppose that $p(z)$ and $q(z)$ are polynomials in z, the degree of p is less than the degree of q, the zeroes of q are all simple, and the zeroes of q are located at the points $a_1, \ldots, a_n$. Prove that
$$\frac{p(z)}{q(z)} = \sum_{j=1}^{n} \frac{\operatorname{Res}_{p/q}(a_j)}{z-a_j} \qquad \text{when } z \in \mathbb{C} \setminus \{a_1, \ldots, a_n\}.$$

4. Find a function that is meromorphic on the whole plane $\mathbb{C}$ with the property that every disc of radius 1 (and arbitrary center) contains at least one pole of the function.