Instructions In each item, either give a concrete example satisfying the conditions or prove that no example exists (whichever is appropriate). Work as many of the 16 items as you wish.

Your score for n correct items is $10\min(n, 8) + 5\max(n - 8, 0)$; that is, your first 8 correct items count 10 points each, and additional correct items count 5 points each. (More than 12 correct items will give you extra credit: namely, a score greater than 100.)

1. A non-empty open set U and a holomorphic function f on U such that there does not exist a holomorphic function g on U with the property that the derivative $g' = f$.

2. A non-empty open set U, a holomorphic function f on U, and a simple closed curve γ in U such that $\int_\gamma f(z)\,dz = \sqrt{2}$.

3. A holomorphic function f on the unit disc $\{\, z \in \mathbb{C} : |z| < 1 \,\}$ such that the nth derivative $f^{(n)}(0) = (n!)^2$ for every positive integer n.

4. A non-polynomial entire function with exactly one zero.

5. A holomorphic function f on the disc $\{\, z \in \mathbb{C} : |z| < 2 \,\}$ such that $f(1/n) = (-1)^n/n$ for every positive integer n.

6. A holomorphic function f on the punctured disc $\{\, z \in \mathbb{C} : 0 < |z| < 2 \,\}$ such that $\int_{|z|=1} f(z)\,dz = 0$, and f has an essential singularity at the origin.

7. A rational function f having a pole at 0 such that the residue of f at 0 equals 2 and the residue of the derivative f' at 0 equals 1.

8. A positive real number a such that $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^4 + a^4}\,dx = 1$.

9. A holomorphic function (not necessarily one-to-one) that maps the unit disc $\{ z \in \mathbb{C} : |z| < 1 \}$ onto the punctured disc $\{ z \in \mathbb{C} : 0 < |z| < 1 \}$.

10. A continuous function f on the closed disc $\{ z \in \mathbb{C} : |z| \leq 1 \}$ that is holomorphic in the open disc, has 617 simple zeroes in the open disc, and satisfies the inequality $|f(z)| \leq 1$ when $|z| \leq 1$.

11. A continuous function on the upper half-plane $\{ z \in \mathbb{C} : \operatorname{Im} z \geq 0 \}$ that is holomorphic in the open half-plane and that satisfies the property $\sup\{ |f(z)| : \operatorname{Im} z > 0 \} \neq \sup\{ |f(z)| : \operatorname{Im} z = 0 \}$.

12. A linear fractional transformation (Möbius transformation) f such that $f(1) = 1$, $f(2) = 4$, $f(3) = 9$, and $f(4) = 16$.

13. A holomorphic function mapping the unit disc $\{ z \in \mathbb{C} : |z| < 1 \}$ into itself such that $f(1/2) = -1/2$, and the derivative $f'(1/2) = 1$.

14. A family of entire functions such that the image of each function is contained in the punctured plane $\mathbb{C} \setminus \{0\}$, and the family is not a normal family.

15. A continuous, real-valued function on the plane $\mathbb{C}$ whose restriction to the upper half-plane $\{ z \in \mathbb{C} : \operatorname{Im} z > 0 \}$ is harmonic and whose restriction to the lower half-plane $\{ z \in \mathbb{C} : \operatorname{Im} z < 0 \}$ is harmonic, but which is not a harmonic function on all of $\mathbb{C}$.

16. A continuous, real-valued function u on the closed unit disc $\{ z \in \mathbb{C} : |z| \leq 1 \}$, harmonic in the open disc, such that $u(1/2) = 3/4$, and $|u(z)| \leq 1$ on the boundary of the disc.