The first problem on the sixth homework assignment asked you to find an example of a sequence $\{a_n\}$ of complex numbers such that the series $\sum_{n=1}^{\infty} a_n$ converges (conditionally), yet the series $\sum_{n=1}^{\infty} a_n^3$ diverges. Here are some remarks about this problem.

First of all, there is no hope of finding an example in which $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. Indeed, the terms of a convergent series must tend to 0, so when n is sufficiently large, it will be the case that $|a_n| \le 1$, whence $|a_n^3| \le |a_n|$. Therefore the series $\sum_{n=1}^{\infty} a_n^3$ will be absolutely convergent if $\sum_{n=1}^{\infty} a_n$ is.

Consequently, one has to seek for the required example among the conditionally convergent series. The most popular example is to set a_n equal to $\dfrac{\exp(2\pi i n/3)}{n^{1/3}}$. Then $a_n^3 = 1/n$, so the series $\sum_{n=1}^{\infty} a_n^3$ is the divergent harmonic series. What remains to check is that the series $\sum_{n=1}^{\infty} a_n$ does converge.

Method 1 In your first calculus class, you probably learned a convergence test for alternating series stating that if $\{b_n\}_{n=1}^{\infty}$ is a sequence of positive real numbers monotonically decreasing to 0, then the series $\sum_{n=1}^{\infty} (-1)^n b_n$ converges. But this test does not fit the situation at hand.

You may or may not have seen a generalization of the alternating-series test that does apply. Namely, Dirichlet's test says that if (as before) $\{b_n\}_{n=1}^{\infty}$ is a sequence of positive real numbers monotonically decreasing to 0, and if $\sum_{n=1}^{\infty} c_n$ is a possibly divergent series of complex numbers having bounded partial sums (that is, $|\sum_{n=1}^{k} c_n|$ stays bounded independently of k), then the series $\sum_{n=1}^{\infty} c_n b_n$ converges. (In the alternating-series test, $c_n = (-1)^n$, so $|\sum_{n=1}^{k} c_n|$ is either 0 or 1, hence bounded independently of k.) Dirichlet's test is present—but not prominent—in the textbook: see Problem 8(a) on page 18 in Chapter 2. (The authors do not name the test.)

To apply the test to solve your problem, set b_n equal to $1/n^{1/3}$ and c_n equal to $\exp(2\pi i n/3)$. Since

$$\exp(2\pi i/3) + \exp(4\pi i/3) + \exp(6\pi i/3) = 0, \tag{1}$$

it follows that each partial sum $\sum_{n=1}^{k} c_n$ is either $\exp(2\pi i/3)$ or $\exp(2\pi i/3) + \exp(4\pi i/3)$ (which simplifies to -1) or 0. Thus these partial sums all have modulus bounded by 1, and Dirichlet's test implies that $\displaystyle\sum_{n=1}^{\infty} \frac{\exp(2\pi i n/3)}{n^{1/3}}$ does converge.

(The proof of Dirichlet's test, incidentally, is based on Abel's technique of partial summation, analogous to integration by parts.)

Method 2 It is possible to verify the convergence by concrete estimation of groups of terms, as follows. The mean-value theorem from real calculus, applied to the function $1/x^{1/3}$, implies that

$$\left|\frac{1}{(n+1)^{1/3}} - \frac{1}{n^{1/3}}\right| \le \frac{1}{3}\cdot\frac{1}{n^{4/3}} \qquad \text{and} \qquad \left|\frac{1}{(n+2)^{1/3}} - \frac{1}{n^{1/3}}\right| \le \frac{2}{3}\cdot\frac{1}{n^{4/3}}. \tag{2}$$

(What is being used here is that the change in a real-valued function on an interval is at most the width of the interval times the maximal value of the absolute value of the derivative. The derivative of $1/x^{1/3}$ is $-(1/3)/x^{4/3}$, and the absolute value of this derivative takes its biggest value on the interval $[n, n+2]$ at the left-hand endpoint.)

In view of equation (1), the expression

$$\left|\frac{\exp(2\pi i n/3)}{n^{1/3}} + \frac{\exp(2\pi i(n+1)/3)}{(n+1)^{1/3}} + \frac{\exp(2\pi i(n+2)/3)}{(n+2)^{1/3}}\right|$$

can be rewritten, by adding and subtracting

$$\frac{\exp(2\pi i(n+1)/3)}{n^{1/3}} + \frac{\exp(2\pi i(n+2)/3)}{n^{1/3}},$$

as

$$\left|\left(\frac{1}{(n+1)^{1/3}} - \frac{1}{n^{1/3}}\right)\exp(2\pi i(n+1)/3) + \left(\frac{1}{(n+2)^{1/3}} - \frac{1}{n^{1/3}}\right)\exp(2\pi i(n+2)/3)\right|.$$

By the triangle inequality and property (2), this expression is bounded above by $1/n^{4/3}$.

To demonstrate convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{\exp(2\pi i n/3)}{n^{1/3}}$, it suffices to show that expressions of the form

$$\left|\sum_{n=j}^{k} \frac{\exp(2\pi i n/3)}{n^{1/3}}\right| \tag{3}$$

get arbitrarily close to 0 when j and k get large. If necessary, add one or two terms to the end of the sum to guarantee that the number of terms is a multiple of 3; the error thereby introduced has modulus less than $2/k^{1/3}$. The estimate on groups of three terms derived above shows that

$$\left|\sum_{n=j}^{k} \frac{\exp(2\pi i n/3)}{n^{1/3}}\right| < \frac{2}{k^{1/3}} + \sum_{n=j}^{\infty} \frac{1}{n^{4/3}}. \tag{4}$$

Since $\sum_{n=1}^{\infty} 1/n^{4/3}$ is a convergent series, both terms on the right-hand side of (4) do indeed get close to 0 when j and k get large. That deduction completes the argument.