Theory of Functions of a Complex Variable I

Instructions Solve **six** of the following seven problems. Please write your solutions on your own paper.

These problems should be treated as essay questions. A problem that says "find" or that asks a question requires an explanation to support the answer. Please explain your reasoning in complete sentences.

1. Find the first two nonzero terms of the Laurent series of $\dfrac{1}{z^2(e^z - e^{-z})}$ that is valid in the punctured disk where $0 < |z| < \pi$.

2. Classify each isolated singularity of the function $\dfrac{1}{z^2(z+1)} + \sin\left(\dfrac{1}{z}\right)$: is the singularity removable? essential? a pole?

3. Use the residue theorem to prove that
$$\frac{1}{2\pi}\int_0^{2\pi} (\sin\theta)^{2n}\, d\theta = \frac{(2n)!}{(n!\,2^n)^2}$$
when n is a natural number.

4. Suppose that f is analytic in an open neighborhood of the closed unit disk $\overline{D}(0,1)$, and $|f(z)| < 1$ when $|z| \le 1$. Brouwer's fixed-point theorem from topology implies that f has at least one fixed point in the closed unit disk (that is, there exists a point z_0 such that $f(z_0) = z_0$). Use Rouché's theorem to show that in this special setting, the function f has *exactly one* fixed point in the closed unit disk.

5. Suppose that f and g are analytic in an open neighborhood of the closed unit disk $\overline{D}(0,1)$, and f has no zeroes on the unit circle $C(0,1)$. Let the distinct zeroes of f in $D(0,1)$ be $a_1, \ldots, a_n$, and suppose that each of these zeroes is simple (that is, first order). Prove that
$$\frac{1}{2\pi i}\int_{C(0,1)} \frac{f'(z)}{f(z)} g(z)\, dz = \sum_{j=1}^{n} g(a_j).$$

6. Find a linear fractional transformation (a Möbius transformation) that fixes the points 1 and -1 and maps i to 0.

7. Does there exist a linear fractional transformation (a Möbius transformation) that maps the open half-disk $\{\, z \in \mathbb{C} : |z| < 1 \text{ and } \operatorname{Im} z > 0 \,\}$ onto the open first quadrant? (See the figure below.) Explain.