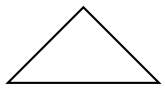
Theory of Functions of a Complex Variable I

Instructions Please solve *six* of the following seven problems. Treat these problems as essay questions: supporting explanation is required.

1. When R is a real number greater than 1, let C_R denote the triangle (oriented counterclockwise) with vertices $-R$, R, and iR. Does the limit



$$\lim_{R\to\infty}\int_{C_R}\frac{1}{1+z^2}\,dz \qquad \text{exist?}$$

2. Let D denote the unit disk, $\{\, z \in \mathbb{C} : |z| < 1 \,\}$. If $f : D \to D$ is a holomorphic function, then how big can $|f''(0)|$ be?

3. Riemann's famous zeta function can be defined as follows:

$$\zeta(z) = \sum_{n=1}^{\infty}\frac{1}{n^z} \qquad \text{when } \operatorname{Re} z > 1.$$

(Recall that n^z means $\exp\{z\ln(n)\}$ when n is a positive integer.) Notice that this infinite series is not a power series. Does this infinite series converge uniformly on each compact subset of the open half-plane $\{\, z \in \mathbb{C} : \operatorname{Re} z > 1 \,\}$?

4. In what region of the complex plane does the integral

$$\int_0^1 \frac{1}{(1-zt)^2}\,dt$$

represent a holomorphic function of z? (The formula is to be understood as an integral in which the real variable t moves along the real axis from 0 to 1.)

5. In what region of the complex plane does the infinite series

$$\sum_{n=1}^{\infty}\left(\frac{1}{z^n}+\frac{z^n}{2^n}\right)$$

represent a holomorphic function of z?

6. There cannot exist an entire function f with the property that

$$f\left(\frac{1}{n}\right) = \frac{(-1)^n}{n^2} \qquad \text{for every positive integer } n.$$

Why not?

7. Prove the following property of the gamma function:

$$\left|\Gamma\left(\tfrac{1}{2}+it\right)\right|^2 = \frac{\pi}{\cosh(\pi t)} \qquad \text{when } t \in \mathbb{R}.$$

Hint: Recall that $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$, and $\cosh(z) = \frac{1}{2}(e^z + e^{-z})$.