Final Examination

Part I State three of the following six items: the Cauchy–Riemann equations; some version of Cauchy's theorem that applies to an annulus; Morera's theorem; Cauchy's estimate for derivatives at the center of a disk; some version of Rouché's theorem; the residue theorem.

Part II Solve three of the following six problems.

- Problem 1 on the August 2008 qualifying exam: Find the Laurent series of $\dfrac{1}{z(z-1)(z-2)}$ valid in the annulus $\{ z \in \mathbb{C} : 1 < |z| < 2 \}$.

- Problem 4 on the January 2009 qualifying exam: Prove that if a is an arbitrary complex number, and n is an integer greater than 1, then the polynomial $1 + z + az^n$ has at least one zero in the disk where $|z| \le 2$.
  Hint: The product of the zeroes of a monic polynomial of degree n equals $(-1)^n$ times the constant term.

- Problem 8 on the January 2009 qualifying exam: Suppose f is holomorphic in the vertical strip where $|\operatorname{Re}(z)| < \pi/4$, and $|f(z)| < 1$ for every z in the strip, and $f(0) = 0$. Prove that $|f(z)| \le |\tan(z)|$ for every z in the strip.

- Problem 2 on the January 2010 qualifying exam: Suppose that f has an isolated singularity at the point a, and f'/f has a first-order pole at a. Prove that f has either a pole or a zero at a.

- Problem 3 on the August 2010 qualifying exam: Calculate the "Fresnel integrals"
  $$\int_0^\infty \sin(x^2)\,dx \qquad \text{and} \qquad \int_0^\infty \cos(x^2)\,dx,$$
  which play an important role in diffraction theory. (You may assume known that $\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$.)

- Problem 6 on the January 2011 qualifying exam: Suppose f is a holomorphic function (not necessarily bounded) on $\{ z \in \mathbb{C} : |z| < 1 \}$, the open unit disk, such that $f(0) = 0$. Prove that the infinite series $\sum_{n=1}^{\infty} f(z^n)$ converges uniformly on compact subsets of the open unit disk.