Final examination

December 16 (Wednesday)

1:00-3:00pm

in the usual classroom

Möbius transformations (linear fractional transformations)

Moebius

$\mathbb{C}\times\mathbb{C} \to \mathbb{C}\times\mathbb{C}$ via linear transformations

Do those maps induce maps of projective space $\mathbb{C}\times\mathbb{C}\setminus\{(0,0)\}/\sim$?

Invertible linear transformations preserve $(0,0)$, so have some hope of inducing maps on $\mathbb{CP}^1$.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on $[z_1 : z_2]$

via $[z_1 : z_2] \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

$$= \begin{pmatrix} az_1 + bz_2 \\ cz_1 + dz_2 \end{pmatrix} \mapsto [az_1+bz_2 : cz_1+dz_2]$$

In the coordinate chart given by $z^{\in\mathbb{C}} \to [z:1]$, such a transformation has the form

$$z \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix} \mapsto$$

$$[az+b : cz+d] = \left[\frac{az+b}{cz+d} : 1\right]$$

$$\mapsto \frac{az+b}{cz+d} \qquad (ad-bc \neq 0)$$

Special cases of Möbius transformations

- $z \mapsto z + b$ is a translation. $(a=1,\ c=0,\ d=1)$
- $z \mapsto az$ is a dilation when $a > 0$. $(b=0,\ c=0,\ d=1)$
- $z \mapsto az$ is a rotation when $|a| = 1$.
- $z \mapsto 1/z$ is inversion. $(a=0,\ b=1,\ c=1,\ d=0)$

These four basic maps generate all Möbius transformations.

$$\frac{az+b}{cz+d} = \frac{a}{c} - \frac{(ad-bc)}{c(cz+d)}$$

These transformations are conformal mappings.

Inversion

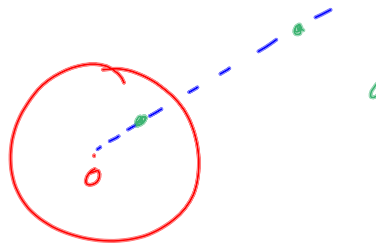
$z \mapsto \frac{1}{z}$ or $re^{i\theta} \mapsto \frac{1}{r}e^{-i\theta}$

$e^{i\theta} \mapsto e^{-i\theta}$ inversion in the unit circle

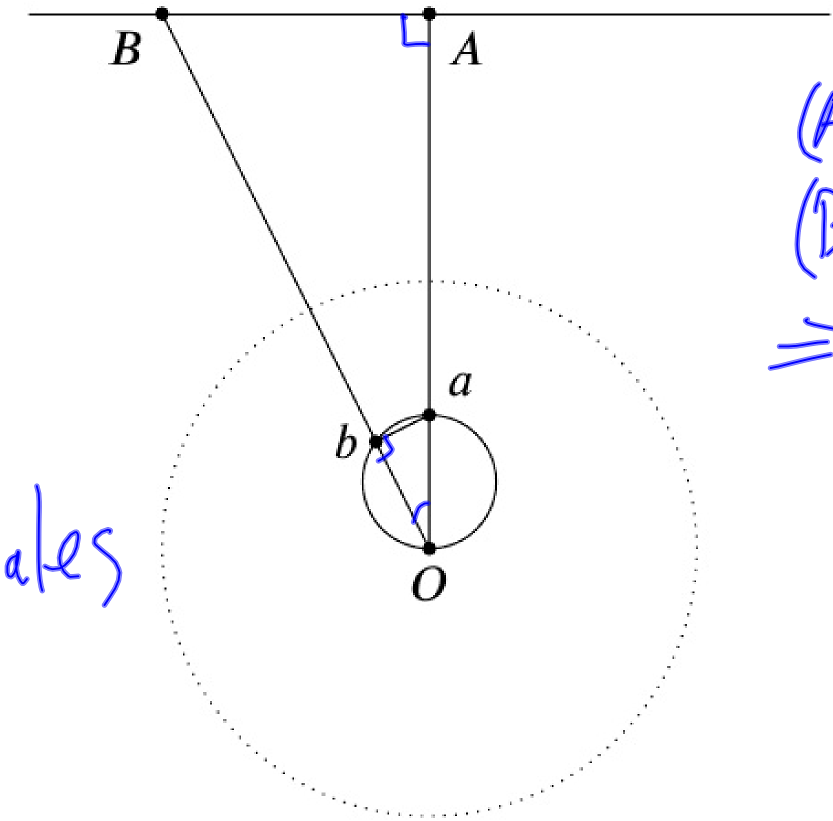
Inversion in a circle of radius R is $z \mapsto \frac{R^2}{z}$.

<u>Geometric</u> inversion is analytic inversion composed with complex conjugation: namely

$z \mapsto R^2/\bar{z}$:



Geometric inversion sends a point to another point on same ray, and product of distances from the center equals square of radius.



Thales

$$(AO)(aO) = \text{radius}^2$$
$$(BO)(bO) = \text{radius}^2$$
$$\Rightarrow \frac{(AO)}{(BO)} = \frac{(bO)}{(aO)}$$

<u>Corollary</u> Möbius transformations preserve the set of lines and circles.

Inversion and stereographic projection