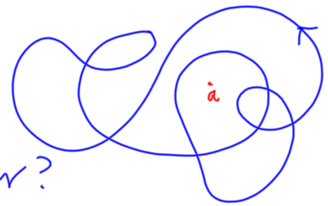
Winding number (or index)

If γ is a closed curve, then $n(\gamma; a) := \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\,dz$

$$= \frac{1}{2\pi i}\int_0^1 \frac{\gamma'(t)}{\gamma(t)-a}\,dt$$



Why is this quantity an integer?

Let $g(s) = \frac{1}{2\pi i}\int_0^s \frac{\gamma'(t)}{\gamma(t)-a}\,dt$.

$2\pi i\, g(s)$ "should be" $\log(\gamma(s)-a) + \text{const}$

except that log doesn't make sense.

$e^{2\pi i g(s)}$ "should be" $(\text{const})(\gamma(s)-a)$.

Let's try to prove that

$(\gamma(s)-a)e^{-2\pi i g(s)}$ is constant.

Aside: if this expression is constant,

then $(\gamma(0)-a)e^{-2\pi i \cancel{g(0)}\, 0} = \gamma(0)-a$

$= (\gamma(1)-a)e^{-2\pi i g(1)}$.

Then $e^{-2\pi i g(1)} = 1$, so $g(1) \in \mathbb{Z}$.

$\hookrightarrow$ Try to show derivative $= 0$.

$\frac{d}{ds}\left[(\gamma(s)-a)e^{-2\pi i g(s)}\right] =$

$\gamma'(s)e^{-2\pi i g(s)} + (\gamma(s)-a)e^{-2\pi i g(s)}(-2\pi i\, g'(s))$

$= \gamma'(s)e^{-2\pi i g(s)} + (\gamma(s)-a)e^{-2\pi i g(s)}$ times (by fundamental thm of calculus) $-2\pi i \frac{1}{2\pi i}\frac{\gamma'(s)}{\gamma(s)-a}$

$= 0$. !

Two observations:

(1) When $a \to \infty$,

$n(\gamma; a) \to 0$ because

the integrand $\dfrac{\gamma'(t)}{\gamma(t) - a}$

tends to zero uniformly with respect to t.

by observation 2, $n(\gamma; a) = 0$ for $\underline{a}$ in the unbounded component of $\mathbb{C} - \gamma$.

(2) $n(\gamma; a) = \dfrac{1}{2\pi i} \displaystyle\int_\gamma \frac{1}{z - a}\, dz$

is continuous with respect to $\underline{a}$ when $\underline{a}$ stays within a component of the complement of γ.

Being integer-valued, $n(\gamma; a)$ is <u>constant</u> on each component of $\mathbb{C} - \gamma$.

Simple connectivity

Suppose G is a connected open set. G is <u>simply connected</u> if for every closed curve γ in G and for every point $a \in \mathbb{C} \setminus G$,

$$n(\gamma; a) = 0.$$

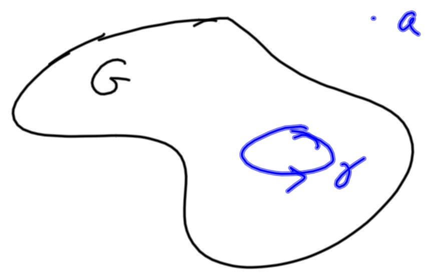
Equivalently, G has no holes in the sense that $(\mathbb{C} \cup \{\infty\}) \setminus G$ is connected.

Yes

No

Proof of equivalence.

(1) Suppose no holes.



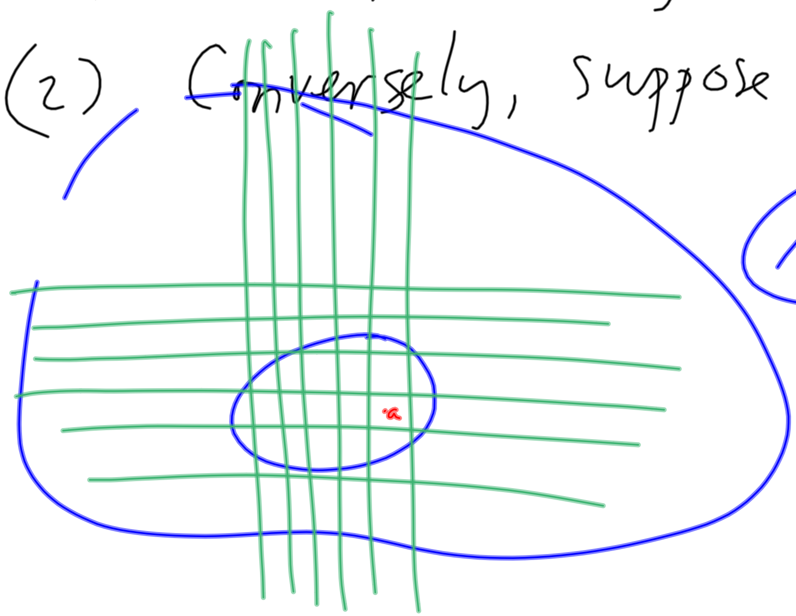
$$\gamma \subset G$$

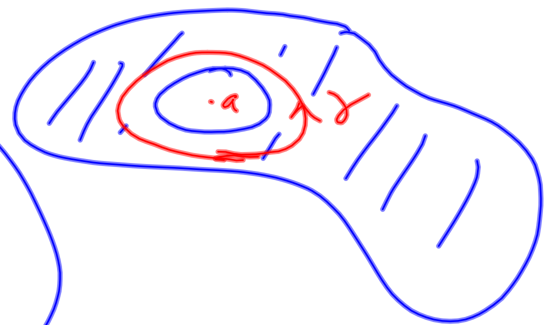
$$\gamma^c \supset G^c \quad \text{so}$$

unbounded component of complement of γ contains every point $\underline{a}$ in $\mathbb{C} - G$.

So $n(\gamma; a) = 0$ by observation 1 above.

(2) Conversely, suppose G has a hole.





Assignment

Read Section 7 of Chapter IV, pages 97-99
(counting zeroes and the open mapping theorem).