**Homology version of Cauchy's theorem**

Winding number $n(\gamma; a) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\,dz.$

If $f$ is analytic in a region, and $\gamma$ is a closed curve in the region whose winding number about every point in the complement of the region is zero, then

$$\int_\gamma f(z)\,dz = 0, \qquad \text{and}$$

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz = n(\gamma; a) f(a)$$

In particular, hypothesis holds for simply connected regions.

$$g(w,z)=\begin{cases}\dfrac{f(z)-f(w)}{z-w} & \text{if } z\neq w\\ f'(w) & \text{if } z=w\end{cases}$$

Check continuity when $(w,z)$ is close to $(0,0)$.

$$f(z)=\sum_{n=0}^{\infty} c_n z^n$$

with some radius of convergence $R$.

If $z\neq w$, then

$$g(w,z)=\frac{\sum_{n=1}^{\infty} c_n(z^n-w^n)}{z-w}$$

$$=\sum_{n=1}^{\infty} c_n\cdot p_n(w,z)$$

where $p_n(w,z)=z^{n-1}+z^{n-2}w+\cdots+w^{n-2}z+w^{n-1}$

Observe $p_n(w,w)=n\,w^{n-1}$, so

$$\sum_{n=1}^{\infty} c_n\, p_n(w,w)=f'(w)=g(w,w).$$

So if $\sum_{n=1}^{\infty} c_n\, p_n(w,z)$ converges uniformly for $z$ and $w$ in a neighborhood of $0$, then $g$ is continuous at $(0,0)$.

Restrict $z, w$ to satisfy $|z|\leq r<R$, $|w|\leq r<R$.

Then $|p_n(w,z)|\leq n\cdot r^{n-1}$

So $\quad |c_n\, p_n(w,z)|\leq |c_n|\,n\,r^{n-1}$

These terms are the terms of an absolutely convergent series: namely, the series for $f'(r)$.

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7. Let $\gamma(t) = 1 + e^{it}$ for $0 \leq t \leq 2\pi$. Find $\int_\gamma \left( \frac{z}{z-1} \right)^n dz$ for all positive integers $n$.

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4. Suppose that $f: G \to \mathbb{C}$ is analytic and one-one; show that $f'(z) \neq 0$ for any $z$ in $G$.

injective

6. Let $P: \mathbb{C} \to \mathbb{R}$ be defined by $P(z) = \operatorname{Re} z$; show that $P$ is an open map but is not a closed map. (Hint: Consider the set $F = \{z: \operatorname{Im} z = (\operatorname{Re} z)^{-1}$ and $\operatorname{Re} z \neq 0\}$.)