Riemann's ζ function

zeta

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \text{when } \mathrm{Re}(s) > 1.$$

Observe $\left| \frac{1}{n^s} \right| = \left| e^{-s \ln(n)} \right|$

$$= e^{-\mathrm{Re}(s) \ln(n)} = \frac{1}{n^{\mathrm{Re}(s)}}.$$

How to extend $\zeta(s)$ to a larger open set?

Consider related series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}.$$

Claim series converges when $\mathrm{Re}(s) > 0$.

When $s \in \mathbb{R}$, the claim follows from alternating series test.

Assume OK for $s \in \mathbb{C}$ for now.

Next observe

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = \sum_{n \text{ odd}} \frac{1}{n^s} - \sum_{n \text{ even}} \frac{1}{n^s}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^s} - 2 \sum_{n \text{ even}} \frac{1}{n^s}$$

$$= \zeta(s) - 2 \sum_{k=1}^{\infty} \frac{1}{(2k)^s}$$

$$= \zeta(s) \left[1 - \frac{2}{2^s} \right]$$

So, when $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$$

This formula gives a way to extend definition of $\zeta(s)$ from $\mathrm{Re}(s) > 1$ to $\mathrm{Re}(s) > 0$.

Riemann hypothesis [unsolved]

All zeroes of $\zeta(s)$ when $0 < \mathrm{Re}(s) < 1$ have real part equal to $\frac{1}{2}$.