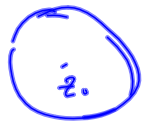
Maximum principle revisited

Suppose f is analytic and nonconstant in some connected open set. Then $|f|$ cannot have a (weak) local maximum.

<u>Proof.</u>



$$\left|f(z_0)\right| \le \frac{1}{2\pi}\int_0^{2\pi} \left|f(z_0+re^{i\theta})\right| d\theta$$

(for all small positive r)

If $|f(z_0)|$ is a local max (weak), then

$$|f(z_0)| = |f(z_0+re^{i\theta})|$$ for all θ and every small r

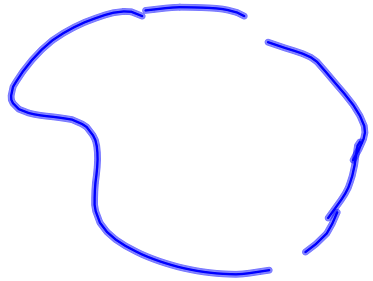
So $|f|$ is constant on a disk.

$|f|^2$ is constant on a disk.

$f\cdot\overline{f}$ is constant

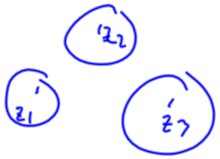
By previous exercise about $f\bar{g}$, it follows that f is constant. Contradiction.

<u>Corollary</u> If f is analytic on a <u>bounded</u> open set, and if f is continuous on the closure, then $|f|$ attains its maximum on the boundary of the set.



Suppose f is a rational function with degree denominator > degree numerator.

Let g = sum of principal parts of the Laurent series near each zero of the denominator of f.



Claim $f = g$, or $f - g \equiv 0$.

Laurent series $\displaystyle\sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$

"principal part" is $\displaystyle\sum_{n<0} \cdots$

$f - g$ has only removable singularities by construction

So $f - g$ is an entire function.

$f(z) - g(z) \longrightarrow 0$ when $|z| \to \infty$

So $|f - g|$ is bounded (and small) outside a big disk, and bounded inside that disk. So $|f - g|$ is bounded.

By Liouville's theorem, $f - g$ is constant.

Since $f - g \to 0$ at infinity, that constant value is 0.

Reminder: Exam 2 takes place Thursday, November 5

- Laurent series, residues
- classification of isolated singularities (removable, poles, essential); Riemann's thm. on removable singularities
- Morera's theorem
- Picard's theorems, Casorati-Weierstrass, Liouville's thm.
- Zeroes are isolated; identity principle
- Mean-value property, maximum principle